Lecture 9 – ME6402, Spring 2025
Lyapunov Stability Theory

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Goals of Lecture 9

- Define Lyapunov stability notions
- Lyapunov Stability Theorems

Additional Reading

- Khalil Chapter 4
- Sastry Chapter 5

Lyapunov Stability Theory

Consider a time invariant system

$$\dot{x} = f(x)$$

and assume equilibrium at $x = 0$, *i.e.* $f(0) = 0$. If the equilibrium of interest is $x^* \neq 0$, let $\tilde{x} = x - x^*$:

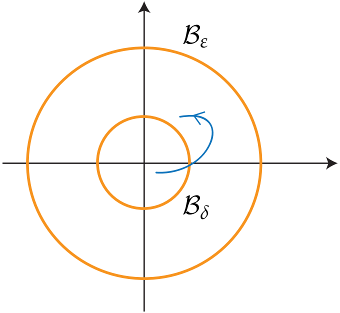
$$\dot{\tilde{x}} = f(x) = f(\tilde{x} + x^*) \triangleq \tilde{f}(\tilde{x}) \Longrightarrow \tilde{f}(0) = 0.$$

- Khalil Chapter 4, Sastry Chapter 5

Lyapunov Stability Theory

Definition: The equilibrium $x = 0$ is stable if for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x(0)| \leq \delta \implies |x(t)| \leq \varepsilon \quad \forall t \geq 0.$$



- Khalil Chapter 4, Sastry Chapter 5

Lyapunov Stability Theory

An equilibrium is unstable if not stable.

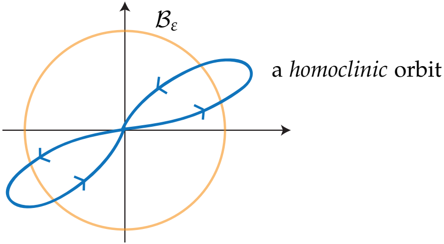
Asymptotically stable if stable and $x(t) \to 0$ for all $x(0)$ in a neighborhood of $x = 0$.

Globally asymptotically stable if stable and $x(t) \to 0$ for every $x(0)$.

- Khalil Chapter 4, Sastry Chapter 5

Lyapunov Stability Theory

Note that $x(t) \to 0$ does not necessarily imply stability: one can construct an example where trajectories converge to the origin, but only after a large detour that violates the stability definition.



- Khalil Chapter 4, Sastry Chapter 5

Lyapunov's Stability Theorem

1. Let D be an open, connected subset of $\mathbb{R}^n$ that includes $x = 0$. If there exists a C^1 function $V : D \to \mathbb{R}$ such that

   $V(0) = 0$ and $V(x) > 0 \quad \forall x \in D - \{0\}$ (positive definite)

   and

   $\dot{V}(x) := \nabla V(x)^T f(x) \leq 0 \quad \forall x \in D$ (negative semidefinite)

   then $x = 0$ is stable.

Aleksandr Lyapunov
(1857-1918)

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2. If $\dot{V}(x) < 0 \quad \forall x \in D - \{0\}$ (negative definite) then $x = 0$ is asymptotically stable.

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2. If $\dot{V}(x) < 0 \quad \forall x \in D - \{0\} \quad$ (negative definite)

   then $x = 0$ is asymptotically stable.

3. If, in addition, $D = \mathbb{R}^n$ and

   $$|x| \to \infty \implies V(x) \to \infty \quad \text{(radially unbounded)}$$

   then $x = 0$ is globally asymptotically stable.

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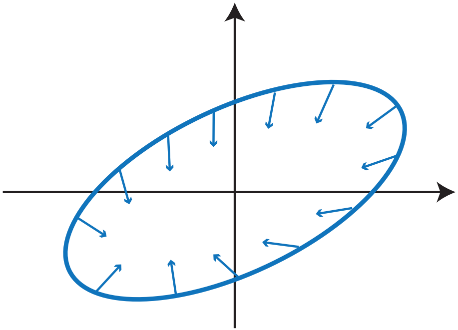
Lyapunov's Stability Theorem (Proof)

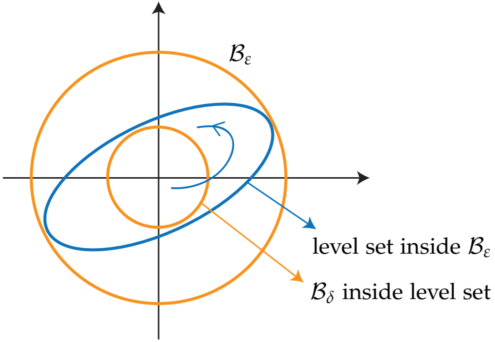
Sketch of the proof:

The sets $\Omega_c \triangleq \{x : V(x) \leq c\}$ for constants c are called *level sets* of V and are positively invariant because $\nabla V(x)^T f(x) \leq 0$.



Lyapunov's Stability Theorem (Proof cont.)

Stability follows from this property: choose a level set inside the ball of radius ε, and a ball of radius δ inside this level set. Trajectories starting in $\mathcal{B}_\delta$ can't leave $\mathcal{B}_\varepsilon$ since they remain inside the level set.



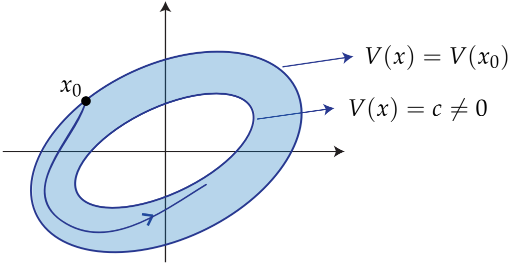
Lyapunov's Stability Theorem (Proof of Asymptotic Stability)

Asymptotic stability:

Since $V(x(t))$ is decreasing and bounded below by 0, we conclude

$$V(x(t)) \to c \geq 0.$$

We will show $c = 0$ (*i.e.*, $x(t) \to 0$) by contradiction. Suppose $c \neq 0$:



- If $\dot{V}(x) < 0 \quad \forall x \in D - \{0\}$ (negative definite) then $x = 0$ is asymptotically stable.

Lyapunov's Stability Theorem (Proof of Asymptotic Stability cont.)

Let

$$\gamma \triangleq \min_{\{x:\ c \le V(x) \le V(x_0)\}} -\dot{V}(x) \ > 0$$

where the minimum exists because it is evaluated over a bounded set, and is positive because $\dot{V}(x) < 0$ away from $x = 0$. Then,

$$\dot{V}(x) \le -\gamma \implies V(x(t)) \le V(x_0) - \gamma t,$$

which implies $V(x(t)) < 0$ for $t > \dfrac{V(x_0)}{\gamma}$, a contradiction because $V \ge 0$. Therefore, $c = 0$ which implies $x(t) \to 0$.

- By positive definiteness of V, the level sets $\{x : V(x) \le \text{constant}\}$ are bounded when the constant is sufficiently small. Since we are proving *local* asymptotic stability we can assume x_0 is close enough to the origin that the constant $V(x_0)$ is sufficiently small.

Lyapunov's Stability Theorem (Proof of Global Asymptotic Stability)

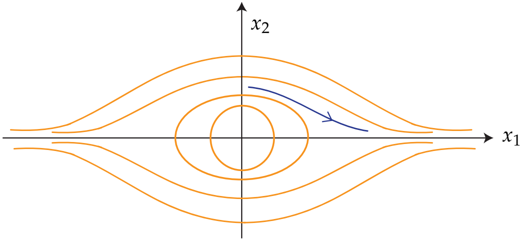
Global asymptotic stability:

Why do we need radial unboundedness?

Example:

$$V(x) = \frac{x_1^2}{1 + x_1^2} + x_2^2$$

Set $x_2 = 0$, let $x_1 \to \infty$: $V(x) \to 1$ (not radially unbounded). Then Ω_c is not a bounded set for $c \geq 1$:



- If, in addition, $D = \mathbb{R}^n$ and is radially unbounded, i.e.,

$$|x| \to \infty \implies V(x) \to \infty$$

then $x = 0$ is globally asymptotically stable.

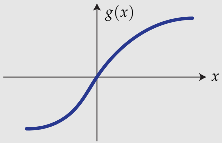
Finding Lyapunov Functions

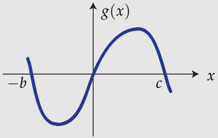
Example:

$$\dot{x} = -g(x) \quad x \in \mathbb{R}, \; xg(x) > 0 \quad \forall x \neq 0$$

$V(x) = \dfrac{1}{2}x^2$ is positive definite and radially unbounded.
$\dot{V}(x) = -xg(x)$ is negative definite. Therefore $x = 0$ is globally asymptotically stable.

If $xg(x) > 0$ only in $(-b, c) - \{0\}$, then take $D = (-b, c)$
$\implies x = 0$ is locally asymptotically stable.
There are other equilibria where $g(x) = 0$, so we know global asymptotic stability is not possible.





Finding Lyapunov Functions

Example:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_2 - g(x_1)a \geq 0,\ xg(x) > 0 \ \ \forall x \in (-b, c) - \{0\}\end{aligned} \tag{1}$$

The choice $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ doesn't work because $\dot{V}(x)$ is sign indefinite (show this).

- The pendulum is a special case with $g(x) = \sin(x)$.

Finding Lyapunov Functions

Example:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -ax_2 - g(x_1)a \geq 0, \; xg(x) > 0 \;\; \forall x \in (-b, c) - \{0\}$$

The choice $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ doesn't work because $\dot{V}(x)$ is sign indefinite (show this). The function

$$V(x) = \int_0^{x_1} g(y)dy + \frac{1}{2}x_2^2$$

is positive definite on $D = (-b, c) - \{0\}$ and

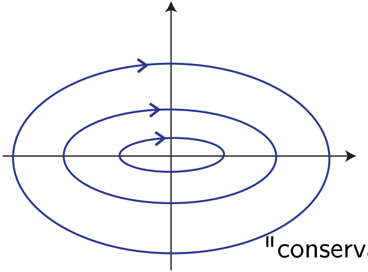
$$\dot{V}(x) = g(x_1)x_2 - ax_2^2 - x_2 g(x_1) = -ax_2^2$$

is negative semidefinite $\implies$ stable.

- The pendulum is a special case with $g(x) = \sin(x)$.

Finding Lyapunov Functions (Example cont.)

If $a = 0$, no asymptotic stability because $\dot{V}(x) = 0 \Longrightarrow V(x(t)) = V(x(0))$.



If $a > 0$, (1) is asymptotically stable but the Lyapunov function above doesn't allow us to reach that conclusion. We need either another V with negative definite $\dot{V}$, or the Lasalle-Krasovskii Invariance Principle to be discussed in the next lecture.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_2 - g(x_1) \\ & a \geq 0,\ xg(x) > 0 \\ & \forall x \in (-b, c) - \{0\} \end{aligned} \qquad (1)$$

- The pendulum is a special case with $g(x) = \sin(x)$.