Lecture 8 – ME6402, Spring 2025
Mathematical Background Continued

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Goals of Lecture 8

- Prove uniqueness and existence theorem of ODEs
- Establish continuous dependence on initial conditions and parameters

Additional Reading

- Khalil Chapter 3
- Sastry Chapter 3.4

Existence and Uniqueness Theorems for ODEs

$$\dot{x} = f(t,x) \quad x(0) = x_0$$

Theorem 1: $f(t,x)$ locally Lipschitz in x and continuous in t
$\Rightarrow$ existence and uniqueness on some finite interval $[0,\delta]$.

- Recall that *local Lipschitz* means for all x_0, there exists L such that

$$|f(x) - f(y)| \leq L|x - y|$$

for all x, y in a neighborhood of x_0.

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Theorem 1: $f(t,x)$ locally Lipschitz in x and continuous in t $\Rightarrow$ existence and uniqueness on some finite interval $[0,\delta]$.

Sketch of the proof: From the local Lipschitz assumption, we can find $r > 0$ and $L > 0$ such that

$$|f(t,x) - f(t,y)| \leq L|x-y| \quad \forall x,y \in \{x \in \mathbb{R}^n : |x - x_0| \leq r\}.$$

From fundamental theorem of calculus, if $x(t)$ is a solution, then:

$$x(t) = \underbrace{x_0 + \int_0^t f(\tau, x(\tau))d\tau}_{=:\, T(x)(t)}.$$

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Sketch of proof (cont.): To apply the Contraction Mapping Theorem:

1. Choose δ small enough that T maps the following subset of $C^n[0,\delta]$ to itself :

$$U = \{x \in C^n[0,\delta] : |x(t) - x_0| \leq r \ \forall t \in [0,\delta]\},$$

i.e.

$$|x(t)-x_0| \leq r \quad \forall t \in [0,\delta] \ \Rightarrow \ |T(x)(t)-x_0| \leq r \quad \forall t \in [0,\delta]. \tag{1}$$

- $$x(t) = x_0 + \underbrace{\int_0^t f(\tau, x(\tau))d\tau.}_{=: T(x)(t)}$$

Sketch of proof (cont.):

1. (Step 1 continued) To find such a δ note that

$$T(x)(t) - x_0 = \int_0^t f(\tau, x(\tau)) d\tau$$
$$= \int_0^t \Big(f(\tau, x(\tau)) - f(\tau, x_0) + f(\tau, x_0) \Big) d\tau$$
$$|T(x)(t) - x_0| \leq \int_0^\delta |f(\tau, x(\tau)) - f(\tau, x_0)| d\tau + \int_0^\delta |f(\tau, x_0)| d\tau$$
$$\leq \int_0^\delta L|x(\tau) - x_0| d\tau + \int_0^\delta h d\tau$$
$$\leq (Lr + h)\delta.$$

Thus, by choosing $\delta \leq \dfrac{r}{Lr+h}$ we ensure that the implication (2) holds.

- $$x(t) = x_0 + \underbrace{\int_0^t f(\tau, x(\tau)) d\tau}_{=: T(x)(t)}.$$
- $$|x(t) - x_0| \leq r \forall t \in [0, \delta] \Rightarrow |T(x)(t) - x_0| \leq r \quad \forall t \in [0, \delta]. \qquad (2)$$
- Let h be a bound on $|f(\tau, x_0)|$

Sketch of proof (cont.):

2. Show that T is a contraction in U, *i.e.*, there exists $\rho < 1$

$$\text{s.t.} \quad x, y \in U \implies |T(x) - T(y)|_C \le \rho |x - y|_C.$$

Note that, for all $t \in [0, \delta]$,

$$|T(x)(t) - T(y)(t)| = \int_0^t |f(\tau, x(\tau)) - f(\tau, y(\tau))| d\tau$$
$$\le L \int_0^t |x(\tau) - y(\tau)| d\tau$$
$$\le \underbrace{L\delta}_{=:\rho} \max_{\tau \in [0,\delta]} |x(\tau) - y(\tau)| = \rho |x - y|_C.$$

Therefore,

$|T(x) - T(y)|_C = \max_{t \in [0,\delta]} |T(x)(t) - T(y)(t)| \le \rho |x - y|_C$ and $\rho < 1$ if $\delta \le r/(Lr + h)$ as prescribed above.

- Recall from Lecture 7: $C^n[0, \delta]$ the set of all continuous functions $[0, \delta] \to \mathbb{R}^n$ with norm

$$|x|_C = \max_{t \in [0,\delta]} |x(t)|$$

Existence for All Time From Global Lipschitzness

Theorem 2: $f(t,x)$ *globally* Lipschitz in x uniformly in t, and continuous in $t \implies$ existence and uniqueness on $[0,\infty)$.

Proof: Choose a δ that doesn't depend on x_0 and apply Theorem 1 repeatedly to cover $[0,\infty)$. This is possible because L works everywhere and we can pick r as large as we wish. Indeed, for any $\delta < \frac{1}{L}$, we can choose r large enough that $\delta \leq \frac{r}{Lr+h}$.

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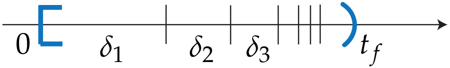
Q: Why can't we do this in Theorem 1?

A: δ depends on x_0 (no universal L) and x_0 changes at the next iteration. We can't use the same δ in every iteration:



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Existence and Uniqueness Theorems for ODEs

- The theorems above are sufficient only, and can be conservative:

Example: $\dot{x} = -x^3$ is not globally Lipschitz but

$$x(t) = \mathrm{sgn}(x_0)\sqrt{\frac{x_0^2}{1+2tx_0^2}}$$

is defined on $[0, \infty)$.

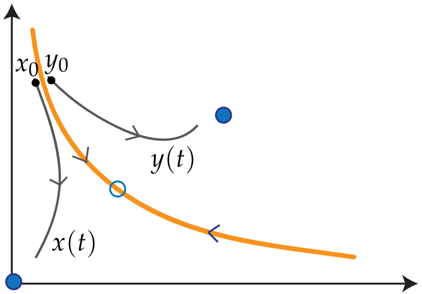
Theorem 3: (Continuous dependence on initial conditions) Let $x(t), y(t)$ be two solutions of $\dot{x} = f(t,x)$ starting from x_0 and y_0, and remaining in a set with Lipschitz constant L on $[0,\tau]$. Then, for any $\varepsilon > 0$, there exists $\delta(\varepsilon,\tau) > 0$ such that

$$|x_0 - y_0| \leq \delta \Longrightarrow |x(t) - y(t)| \leq \varepsilon \quad \forall t \in [0,\tau].$$

- This conclusion does not hold on infinite time intervals (even if f is globally Lipschitz).

Example

Example: bistable system



If ε is smaller than the distance between the two stable equilibria, no choice of δ guarantees $|x(t) - y(t)| \leq \varepsilon \quad \forall t \geq 0$.

Continuous Dependence on Initial Conditions and Parameter (cont.)

- Theorem 3 also shows continuous dependence on parameter μ in $f(t,x,\mu)$ if we rewrite the system equations as:

$$\begin{aligned}\dot{x} &= f(t,x,\mu) \\ \dot{\mu} &= 0\end{aligned} \qquad X = \begin{bmatrix} x \\ \mu \end{bmatrix} \quad \dot{X} = F(t,X) \triangleq \begin{bmatrix} f(t,x,\mu) \\ 0 \end{bmatrix},$$

  where μ appears as a state variable with initial condition $\mu(0) = \mu$.

Q: How do you reconcile bifurcations with continuous dependence on parameters? We could pick two values of the bifurcation parameter arbitrarily close, but one below and one above the critical value, thereby expecting a drastic difference in the solutions.

- A: The two solutions are close in the short term (Theorem 3 holds on finite time intervals); the drastic difference builds up over time.

Sensitivity to Parameters

Consider the system

$$\dot{x} = f(t, x, \mu) \quad x \in \mathbb{R}^n, \mu \in \mathbb{R}^p \tag{3}$$

where μ is a vector of p parameters, and let $\phi(t, x_0, \mu)$ denote the trajectories starting at the initial condition x_0.

To determine to what extent this trajectory depends on the parameters we define the $n \times p$ *sensitivity matrix*:

$$S(t, x_0, \mu) := \frac{\partial \phi(t, x_0, \mu)}{\partial \mu} = \left[\frac{\partial \phi(t, x_0, \mu)}{\partial \mu_1} \cdots \frac{\partial \phi(t, x_0, \mu)}{\partial \mu_p} \right],$$

where each column is the sensitivity with respect to a particular parameter.

Sensitivity to Parameters (cont.)

To see how $S(t,x_0,\mu)$ can be computed numerically, first note that $\phi(t,x_0,\mu)$ satisfies the equation (4), that is,

$$\frac{\partial\phi(t,x_0,\mu)}{\partial t} = f(t,\phi(t,x_0,\mu),\mu).$$

Next, differentiate both sides with respect to μ:

$$\frac{\partial^2\phi(t,x_0,\mu)}{\partial t\partial\mu} = \frac{\partial f}{\partial x}(t,\phi(t,x_0,\mu),\mu)\frac{\partial\phi(t,x_0,\mu)}{\partial\mu} + \frac{\partial f}{\partial\mu}(t,\phi(t,x_0,\mu),\mu)$$

and use the definition of the sensitivity matrix to rewrite this as

$$\frac{\partial S(t,x_0,\mu)}{\partial t} = \frac{\partial f}{\partial x}(t,\phi(t,x_0,\mu),\mu)S(t,x_0,\mu) + \frac{\partial f}{\partial\mu}(t,\phi(t,x_0,\mu),\mu).$$

$$S(t,x_0,\mu) := \frac{\partial\phi(t,x_0,\mu)}{\partial\mu} = \left[\frac{\partial\phi(t,x_0,\mu)}{\partial\mu_1}\cdots\frac{\partial\phi(t,x_0,\mu)}{\partial\mu_p}\right],$$

- Time-varying system with parameters:

$$\dot{x} = f(t,x,\mu) \quad x\in\mathbb{R}^n, \mu\in\mathbb{R}^p \tag{4}$$

Sensitivity to Parameters (cont.)

Thus, S can be computed by numerical integration of (4) simultaneously with

$$\dot{S} = \frac{\partial f}{\partial x}(t,x,\mu)S + \frac{\partial f}{\partial \mu}(t,x,\mu).$$

The initial condition for S is $\frac{\partial x_0}{\partial \mu} = 0$, assuming that x_0 is independent of the parameters.

$$S(t,x_0,\mu) := \frac{\partial \phi(t,x_0,\mu)}{\partial \mu}$$

$$= \left[\frac{\partial \phi(t,x_0,\mu)}{\partial \mu_1} \cdots \frac{\partial \phi(t,x_0,\mu)}{\partial \mu_p} \right],$$

- Time-varying system with parameters:

  $\dot{x} = f(t,x,\mu) \quad x \in \mathbb{R}^n, \mu \in \mathbb{R}^p$

Example

Example: For the harmonic oscillator

$$\begin{aligned}\dot{x}_1 &= -\mu x_2\\ \dot{x}_2 &= \mu x_1\end{aligned}$$

we have

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & -\mu \\ \mu & 0 \end{bmatrix} \qquad \frac{\partial f}{\partial \mu} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}.$$

Thus the sensitivity equation is

$$\dot{S} = \begin{bmatrix} 0 & -\mu \\ \mu & 0 \end{bmatrix} S + \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}.$$