Lecture 7 – ME6402, Spring 2025
Mathematical Background

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Goals of Lecture 7

- Existence and uniqueness of ODEs
- Lipschitz continuity
- Normed linear spaces
- Fixed point theorems
- Contraction mappings

Additional Reading

- Sastry, Chapter 3
- Khalil, Chapter 3 and Appendix B

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Mathematical Background

$$\dot{x} = f(x)$$
$$x(0) = x_0$$

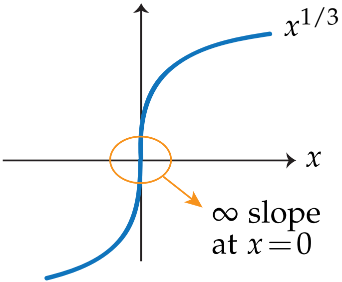
Do solutions exist? Are they unique?

- If $f(\cdot)$ is continuous (C^0) then a solution exists, but C^0 is not sufficient for uniqueness.

Example: Continuity Does not Imply Uniqueness

Example: $\dot{x} = x^{\frac{1}{3}}$ with $x(0) = 0$

$$x(t) \equiv 0, \; x(t) = \left(\frac{2}{3}t\right)^{\frac{3}{2}} \text{ are both solutions}$$



Lipschitz Implies Uniqueness

- Sufficient condition for uniqueness: "Lipschitz continuity" (more restrictive than C^0)

$$|f(x) - f(y)| \leq L|x - y| \qquad (*)$$

<u>Definition:</u> $f(\cdot)$ is *locally Lipschitz* if every point x^0 has a neighborhood where (*) holds for all x, y in this neighborhood for some L.

Example

Example: $(\cdot)^{\frac{1}{3}}$ is NOT locally Lipschitz (due to ∞ slope)

$(\cdot)^3$ *is* locally Lipschitz:

$$x^3 - y^3 = \underbrace{(x^2 + xy + y^2)}_{\substack{\text{in any nbhd} \\ \text{of } x^0\text{, we can} \\ \text{find } L \text{ to upper} \\ \text{bound this}}} (x - y)$$

$$\implies |x^3 - y^3| \leq L|x - y|$$

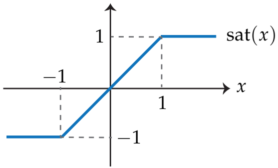
Lipschitz vs. C^1

- If $f(\cdot)$ is continuously differentiable (C^1), then it is locally Lipschitz.

<u>Examples:</u> x^3, x^2, e^x, etc.

The converse is not true: local Lipschitz $\not\Rightarrow C^1$

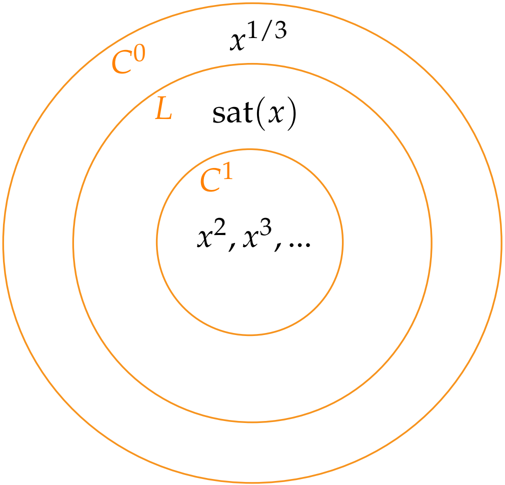
<u>Example:</u>



Not differentiable at $x = \pm 1$, but locally Lipschitz:

$$|\,\text{sat}(x) - \text{sat}(y)| \leq |x - y| \qquad (L = 1).$$

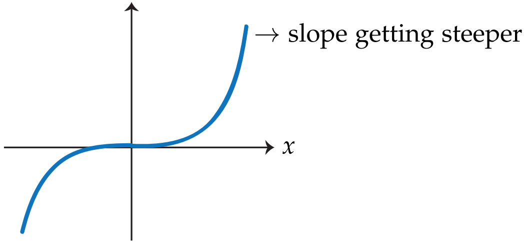
$C^1 \implies$ Lipschitz $\implies C^0$



Global Lipschitz Continuity

Definition continued: $f(\cdot)$ is *globally Lipschitz* if (*) holds $\forall x, y \in \mathbb{R}^n$ (*i.e.*, the same L works everywhere).

Examples: $\text{sat}(\cdot)$ is globally Lipschitz. $(\cdot)^3$ is not globally Lipschitz:



- Suppose $f(\cdot)$ is C^1. Then it is globally Lipschitz iff $\dfrac{\partial f}{\partial x}$ is bounded.

$$L = \sup_x |f'(x)|$$

$$|f(x) - f(y)| \leq L|x - y| \qquad (*)$$

Preview of Existence Theorems

1. $f(\cdot)$ is $C^0 \implies$ existence of solution $x(t)$ on finite interval $[0, t_f)$.
2. $f(\cdot)$ locally Lipschitz $\implies$ existence and uniqueness on $[0, t_f)$.
3. $f(\cdot)$ globally Lipschitz $\implies$ existence and uniqueness on $[0, \infty)$.

Preview of Existence Theorems (cont.)

Examples:

- $\dot{x} = x^2$ (locally Lipschitz) admits unique solution on $[0, t_f)$, but $t_f < \infty$ from Lecture 1 (finite escape).
- $\dot{x} = Ax$ globally Lipschitz, therefore no finite escape

$$|Ax - Ay| \leq L|x - y| \quad \text{with} \quad L = \|A\|$$

The rest of the lecture introduces concepts that are used in proving the existence theorems mentioned above.

Normed Linear Spaces

Definition: $\mathbb{X}$ is a normed linear space (also called normed vector space) if there exists a real-valued norm $|\cdot|$ satisfying:

1. $|x| \geq 0 \quad \forall x \in \mathbb{X}, \quad |x| = 0$ iff $x = 0$.
2. $|x+y| \leq |x| + |y| \quad \forall x, y \in \mathbb{X}$ (triangle inequality)
3. $|\alpha x| = |\alpha| \cdot |x| \quad \forall \alpha \in \mathbb{R}$ and $x \in \mathbb{X}$.

Normed Linear Spaces

Definition: A sequence $\{x_k\}$ in $\mathbb{X}$ is said to be a Cauchy sequence if

$$|x_k - x_m| \to 0 \text{ as } k, m \to \infty.$$

Every convergent sequence is Cauchy. The converse is not true.

Definition: $\mathbb{X}$ is a Banach space if every Cauchy sequence converges to an element in $\mathbb{X}$.

All Euclidean spaces are Banach spaces.

Example

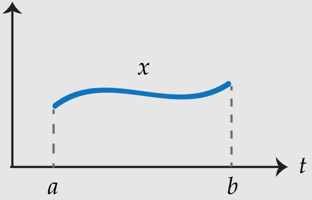
Example:

$C^n[a,b]$: the set of all continuous functions $[a,b] \to \mathbb{R}^n$ with norm:

$$|x|_C = \max_{t\in[a,b]} |x(t)|$$

1. $|x|_C \geq 0$ and $|x|_C = 0$ iff $x(t) \equiv 0$.
2. $|x+y|_C = \max_{t\in[a,b]} |x(t)+y(t)| \leq \max_{t\in[a,b]} \{|x(t)|+|y(t)|\} \leq |x|_C + |y|_C$
3. $|\alpha \cdot x|_C = \max_{t\in[a,b]} |\alpha|\cdot|x(t)| = |\alpha|\cdot|x|_C$

It can be shown that $C^n[a,b]$ is a Banach space.



- Normed linear spaces:

1. $|x| \geq 0 \quad \forall x \in \mathbb{X}, \quad |x| = 0$ iff $x = 0$.
2. $|x+y| \leq |x|+|y| \quad \forall x,y \in \mathbb{X}$ (triangle inequality)
3. $|\alpha x| = |\alpha|\cdot|x| \quad \forall \alpha \in \mathbb{R}$ and $x \in \mathbb{X}$.

Fixed Point Theorems

$$T(x) = x$$

Brouwer's Theorem (Euclidean spaces):
If U is a closed, bounded, convex subset of a Euclidean space and $T : U \to U$ is continuous, then T has a fixed point in U.

Schauder's Theorem (Brouwer's Thm $\to$ Banach spaces):
If U is a closed bounded convex subset of a Banach space $\mathbb{X}$ and $T : U \to U$ is *completely continuous*, then T has a fixed point in U.

- *Completely continuous* means continuous and for any bounded set $B \subseteq U$ the closure of $T(B)$ is compact

Fixed Point Theorems

Contraction Mapping Theorem:

If U is a closed subset of a Banach space and $T : U \to U$ is such that

$$|T(x) - T(y)| \leq \rho |x - y| \quad \rho < 1 \quad \forall x, y \in U$$

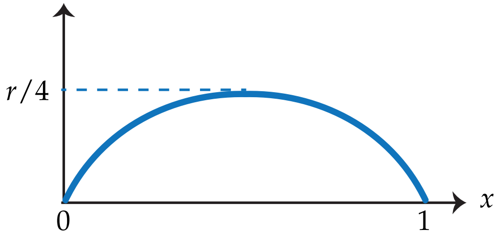
then T has a unique fixed point in U and the solutions of $x_{n+1} = T(x_n)$ converge to this fixed point from any $x_0 \in U$.

Contraction Mapping Example

Example: The logistic map (Lecture 5)

$$T(x) = rx(1-x)$$

with $0 \leq r \leq 4$ maps $U = [0,1]$ to U. $|T'(x)| \leq r \quad \forall x \in [0,1]$, so the contraction property holds with $\rho = r$.

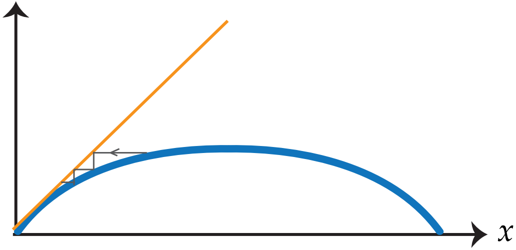


- Contraction property:

$$|T(x) - T(y)| \leq \rho |x - y| \quad \forall x, y \in U$$

Contraction Mapping Example (cont.)

If $r < 1$, the contraction mapping theorem predicts a unique fixed point that attracts all solutions starting in $[0, 1]$.



- Logistic Map:

$$T(x) = rx(1-x)$$

- Contraction mapping theorem condition:

$$|T(x) - T(y)| \leq \rho |x - y|$$

$$\rho < 1, \quad \forall x, y \in U$$

Contraction Mapping Theorem Proof

Proof steps for the Contraction Mapping Thm:

1. Show that $\{x_n\}$ formed by $x_{n+1} = T(x_n)$ is a Cauchy sequence. Since we are in a Banach space, this implies a limit x^* exists.
2. Show that $x^* = T(x^*)$.
3. Show that x^* is unique.

Contraction Mapping Theorem:

If U is a closed subset of a Banach space and $T : U \to U$ is such that

$$|T(x) - T(y)| \leq \rho |x - y|$$
$$\rho < 1 \quad \forall x, y \in U$$

then T has a unique fixed point in U and the solutions of $x_{n+1} = T(x_n)$ converge to this fixed point from any $x_0 \in U$.

Contraction Mapping Theorem Proof

Details of each step:

1.
$$
\begin{aligned}
|x_{n+1} - x_n| = |T(x_n) - T(x_{n-1})| &\leq \rho |x_n - x_{n-1}| \\
&\leq \rho^2 |x_{n-1} - x_{n-2}| \\
&\;\vdots \\
&\leq \rho^n |x_1 - x_0|.
\end{aligned}
$$

$$
\begin{aligned}
|x_{n+r} - x_n| &\leq |x_{n+r} - x_{n+r-1}| + \cdots + |x_{n+1} - x_n| \\
&\leq (\rho^{n+r} + \cdots + \rho^n)|x_1 - x_0| \\
&= \rho^n (1 + \cdots + \rho^r)|x_1 - x_0| \\
&\leq \rho^n \frac{1}{1-\rho} |x_1 - x_0|
\end{aligned}
$$

Since $\frac{\rho^n}{1-\rho} \to 0$ as $n \to \infty$, we have $|x_{n+r} - x_n| \to 0$.

1. Show that $\{x_n\}$ formed by $x_{n+1} = T(x_n)$ is a Cauchy sequence. Since we are in a Banach space, this implies a limit x^* exists.
2. Show that $x^* = T(x^*)$.
3. Show that x^* is unique.

Contraction Mapping Theorem Proof

<u>Details of each step:</u>

2.
$$
\begin{aligned}
|x^* - T(x^*)| &= |x^* - x_n + T(x_{n-1}) - T(x^*)| \\
&\leq |x^* - x_n| + |T(x_{n-1}) - T(x^*)| \\
&\leq |x^* - x_n| + \rho |x^* - x_{n-1}|.
\end{aligned}
$$

Since $\{x_n\}$ converges to x^*, we can make this upper bound arbitrarily small by choosing n sufficiently large. This means that $|x^* - T(x^*)| = 0$, hence $x^* = T(x^*)$.

3. Suppose $y^* = T(y^*) \;\; y^* \neq x^*$.

$$|x^* - y^*| = |T(x^*) - T(y^*)| \leq \rho |x^* - y^*| \implies x^* = y^*.$$

Thus we have a contradiction.

1. Show that $\{x_n\}$ formed by $x_{n+1} = T(x_n)$ is a Cauchy sequence. Since we are in a Banach space, this implies a limit x^* exists.
2. Show that $x^* = T(x^*)$.
3. Show that x^* is unique.