Lecture 6 – ME6402, Spring 2025
Center Manifold Theory and Chaos in Discrete-Time

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Goals of Lecture 6

- Center Manifold Theory
- Discrete-time Systems
- Chaos in Discrete-time

Additional Reading

- Khalil, Chapter 8.1
- Sastry, Chapter 7.6.1

Center Manifold Theory

$$\dot{x} = f(x) \qquad f(0) = 0$$

Suppose $A \triangleq \left. \frac{\partial f}{\partial x} \right|_{x=0}$ has k eigenvalues will zero real parts, and $m = n - k$ eigenvalues with negative real parts.

Define $\begin{bmatrix} y \\ z \end{bmatrix} = Tx$ such that

$$TAT^{-1} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

where

- eigenvalues of A_1 have zero real parts, and
- eigenvalues of A_2 have negative real parts.

Center Manifold Theory

Rewrite $\dot{x} = f(x)$ in the new coordinates:

$$\dot{y} = A_1 y + g_1(y,z)$$
$$\dot{z} = A_2 z + g_2(y,z)$$

where

- $g_i(0,0) = 0$,
- $\dfrac{\partial g_i}{\partial y}(0,0) = 0$,
- $\dfrac{\partial g_i}{\partial z}(0,0) = 0, \; i = 1,2.$

$$\begin{bmatrix} y \\ z \end{bmatrix} = Tx$$

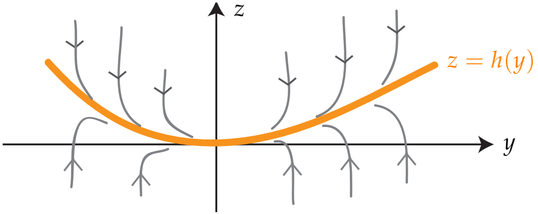
$$TAT^{-1} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

- Eigenvalues of A_1 have zero real parts
- Eigenvalues of A_2 have negative real parts

Center Manifold Theory

<u>Theorem 1:</u> There exists an invariant manifold $z = h(y)$ defined in a neighborhood of the origin such that

$$h(0) = 0 \qquad \frac{\partial h}{\partial y}(0) = 0.$$



$z = h(y)$ is called a *center manifold* in this case.

Reduced System: $\dot{y} = A_1 y + g_1(y, h(y)) \quad y \in \mathbb{R}^k$

- Rewrite $\dot{x} = f(x)$ in the new coordinates:

$$\dot{y} = A_1 y + g_1(y,z)$$
$$\dot{z} = A_2 z + g_2(y,z)$$

$$g_i(0,0) = 0,$$
$$\frac{\partial g_i}{\partial y}(0,0) = 0,$$
$$\frac{\partial g_i}{\partial z}(0,0) = 0, \ i = 1,2.$$

Center Manifold Theory

Theorem 2: If $y = 0$ is asymptotically stable (resp., unstable) for the reduced system, then $x = 0$ is asymptotically stable (resp., unstable) for the full system $\dot{x} = f(x)$.

- Reduced System: $\dot{y} = A_1 y + g_1(y, h(y)) \quad y \in \mathbb{R}^k$

Characterizing the Center Manifold

Define $w \triangleq z - h(y)$ and note that it satisfies

$$\dot{w} = A_2 z + g_2(y,z) - \frac{\partial h}{\partial y}\Big(A_1 y + g_1(y,z)\Big).$$

The invariance of $z = h(y)$ means that $w = 0$ implies $\dot{w} = 0$. Thus, the expression above must vanish when we substitute $z = h(y)$:

$$A_2 h(y) + g_2(y, h(y)) - \frac{\partial h}{\partial y}\Big(A_1 y + g_1(y, h(y))\Big) = 0.$$

To find $h(y)$ solve this partial differential equation for h as a function on y.

Characterizing the Center Manifold

If the exact solution is unavailable, an approximation might be sufficient.

For scalar y, expand $h(y)$ as

$$h(y) = h_2 y^2 + \cdots + h_p y^p + O(y^{p+1})$$

where $h_1 = h_0 = 0$ because $h(0) = \frac{\partial h}{\partial y}(0) = 0$. The notation $O(y^{p+1})$ refers to the higher order terms of power $p+1$ and above.

Example

Example:

$$\dot{y} = yz$$
$$\dot{z} = -z + ay^2 \quad a \neq 0$$

This is of the form at right with $g_1(y,z) = yz$, $g_2(y,z) = ay^2$, $A_2 = -1$. Thus $h(y)$ must satisfy

$$-h(y) + ay^2 - \frac{\partial h}{\partial y} yh(y) = 0.$$

Try $h(y) = h_2 y^2 + O(y^3)$:

- Transformed system:

$$\dot{y} = A_1 y + g_1(y,z)$$
$$\dot{z} = A_2 z + g_2(y,z)$$

- h must satisfy:

$$A_2 h(y) + g_2(y, h(y)) - \frac{\partial h}{\partial y}\Big(A_1 y + g_1(y, h(y))\Big) = 0$$

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$$-h(y) + ay^2 - \frac{\partial h}{\partial y} yh(y) = 0.$$

Try $h(y) = h_2y^2 + O(y^3)$:

$$0 = -h_2y^2 + O(y^3) + ay^2 - (2h_2y + O(y^2))y(h_2y^2 + O(y^3))$$
$$= (a - h_2)y^2 + O(y^3)$$
$$\implies h_2 = a$$

Reduced System: $\dot{y} = y(ay^2 + O(y^3)) = ay^3 + O(y^4)$.

If $a < 0$, the full systems is asymptotically stable. If $a > 0$ unstable.

- Transformed system:

$$\dot{y} = A_1y + g_1(y,z)$$
$$\dot{z} = A_2z + g_2(y,z)$$

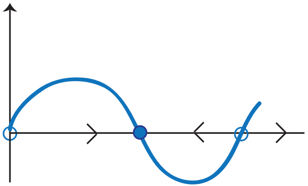
- h must satisfy:

$$A_2h(y) + g_2(y,h(y)) - \frac{\partial h}{\partial y}\Big(A_1y + g_1(y,h(y))\Big) = 0$$

Discrete-Time Models and a Chaos Example

CT: $\dot{x}(t) = f(x(t))$

$f(x^*) = 0$



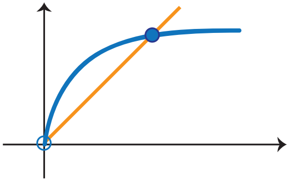
Asymptotic stability criterion:

$\Re\lambda_i(A) < 0$ where $A \triangleq \left.\dfrac{\partial f}{\partial x}\right|_{x=x^*}$

$f'(x^*) < 0$ for first order system

DT: $x_{n+1} = f(x_n) \quad n = 0, 1, 2, \ldots$

$f(x^*) = x^*$ ("fixed point")



Asymptotic stability criterion:

$|\lambda_i(A)| < 1$ where $A \triangleq \left.\dfrac{\partial f}{\partial x}\right|_{x=x^*}$

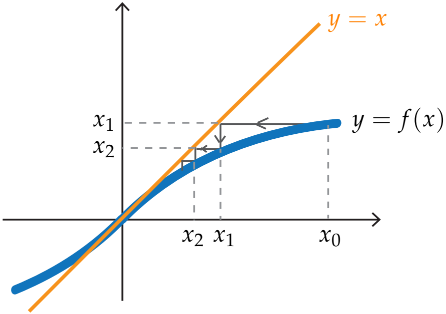
$|f'(x^*)| < 1$ for first order system

Cobweb Diagrams for First Order Discrete-Time Systems

These criteria are inconclusive if the respective inequality is not strict, but for first order systems we can determine stability graphically using a *cobweb diagram*

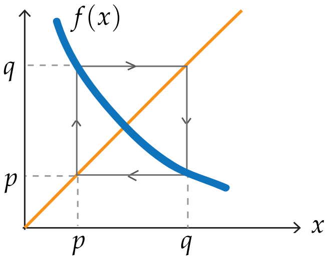
Cobweb Diagrams for First Order Discrete-Time Systems

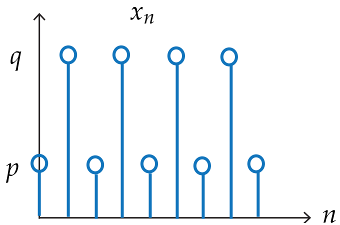
Example: $x_{n+1} = \sin(x_n)$ has unique fixed point at 0. Stability test above inconclusive since $f'(0) = 1$. However, the "cobweb" diagram below illustrates the convergence of iterations to 0:



Oscillations in Discrete-Time Systems

In discrete time, even first order systems can exhibit oscillations:





Detecting Cycles Analytically

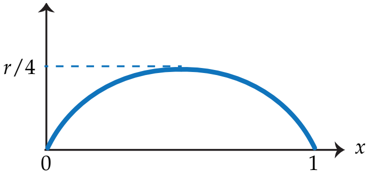
$$f(p) = q \quad f(q) = p \quad \implies \quad f(f(p)) = p \quad f(f(q)) = q$$

- For the existence of a period-2 cycle, the map $f(f(\cdot))$ must have two fixed points in addition to the fixed points of $f(\cdot)$.
- Period-3 cycles: fixed points of $f(f(f(\cdot)))$.

Chaos in a Discrete Time Logistic Growth Model

$$x_{n+1} = r(1 - x_n)x_n$$

Range of interest: $0 \leq x \leq 1 \quad (x_n > 1 \;\Rightarrow\; x_{n+1} < 0)$

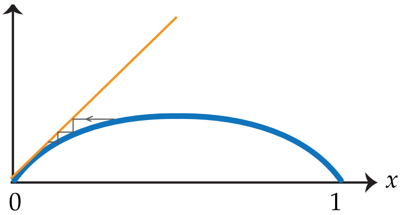


Chaos in a Discrete Time Logistic Growth Model

We will study the range $0 \leq r \leq 4$ so that $f(x) = r(1-x)x$ maps $[0,1]$ onto itself.

Fixed points: $x = r(1-x)x \;\Rightarrow\; \begin{cases} x^* = 0 \quad \text{and} \\ x^* = 1 - \dfrac{1}{r} \text{ if } r > 1. \end{cases}$

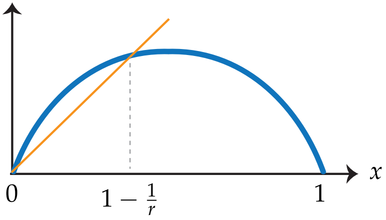
$\underline{r \leq 1}$: $x^* = 0$ unique and stable fixed point



$$x_{n+1} = r(1-x_n)x_n$$

Chaos in a Discrete Time Logistic Growth Model

<u>$r > 1$:</u> $x = 0$ unstable because $f'(0) = r > 1$



$x_{n+1} = r(1 - x_n)x_n$

Chaos in a Discrete Time Logistic Growth Model

Note that a transcritical bifurcation occurred at $r = 1$, creating the new equilibrium

$$x^* = 1 - \frac{1}{r}.$$

Evaluate its stability using $f'(x^*) = r(1 - 2x^*) = 2 - r$.

$$r < 3 \;\Rightarrow\; |f'(x^*)| < 1 \text{ (stable)}$$

$$r > 3 \;\Rightarrow\; |f'(x^*)| > 1 \text{ (unstable)}.$$

$$x_{n+1} = r(1 - x_n)x_n$$

Chaos in a Discrete Time Logistic Growth Model

At $r = 3$, a period-2 cycle is born:

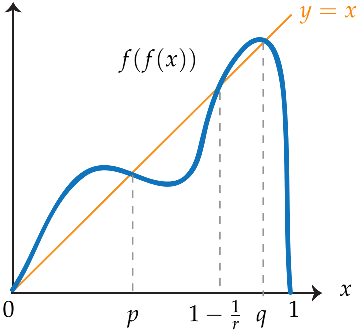
$$
\begin{aligned}
x &= f(f(x)) \\
&= r(1 - f(x))f(x) \\
&= r(1 - r(1-x)x)r(1-x)x \\
&= r^2x(1-x)(1 - r + rx - rx^2) \\
0 &= r^2x(1-x)(1 - r + rx - rx^2) - x
\end{aligned}
$$

Factor out x and $(x - 1 + \frac{1}{r})$, find the roots of the quotient:

$$p, q = \frac{r + 1 \mp \sqrt{(r-3)(r+1)}}{2r}$$

$$x_{n+1} = r(1 - x_n)x_n$$

Chaos in a Discrete Time Logistic Growth Model



$$x_{n+1} = r(1 - x_n)x_n$$

$$p, q = \frac{r + 1 \mp \sqrt{(r-3)(r+1)}}{2r}$$

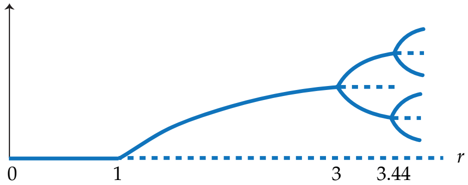
Chaos in a Discrete Time Logistic Growth Model

This period-2 cycle is stable when $r < 1 + \sqrt{6} = 3.4494$:

$$\frac{d}{dx} f(f(x))\bigg|_{x=p} = f'(f(p))f'(p) = f'(p)f'(q) = 4 + 2r - r^2$$

$$|4 + 2r - r^2| < 1 \;\Rightarrow\; 3 < r < 1 + \sqrt{6} = 3.4494$$

At $r = 3.4494$, a period-4 cycle is born!



"period doubling bifurcations"

$$x_{n+1} = r(1 - x_n)x_n$$

Chaos in a Discrete Time Logistic Growth Model

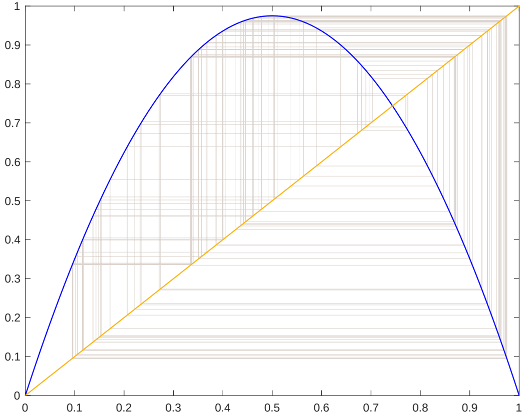
$$x_{n+1} = r(1 - x_n)x_n$$

$r_1 = 3$	period-2 cycle born
$r_2 = 3.4494$	period-4 cycle born
$r_3 = 3.544$	period-8 cycle born
$r_4 = 3.564$	period-16 cycle born
$\vdots$	
$r_\infty = 3.5699$	

After $r > r_\infty$, chaotic behavior for a window of r, followed by windows of periodic behavior (*e.g.*, period-3 cycle around $r = 3.83$).

Chaos in a Discrete Time Logistic Growth Model

Below is the cobweb diagram for $r = 3.9$ which is in the chaotic regime:



$$x_{n+1} = r(1 - x_n)x_n$$