Lecture 5 – ME6402, Spring 2025
Bifurcations

Maegan Tucker

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Goals of Lecture 5

- Understand bifurcations in nonlinear systems

Additional Reading

- Khalil, Chapter 2.7

Bifurcations

A *bifurcation* is an abrupt change in qualitative behavior as a parameter is varied. Examples: equilibria or limit cycles appearing/disappearing, becoming stable/unstable.

Fold Bifurcation

Also known as "saddle node" or "blue sky" bifurcation.

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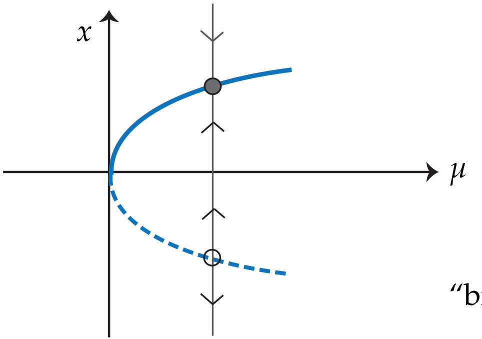

"bifurcation diagram"

Transcritical Bifurcation

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Equilibria: $x = 0$ and $x = \mu$.
$$\frac{\partial f}{\partial x} = \mu - 2x = \begin{cases} \mu & \text{if } x = 0 \\ -\mu & \text{if } x = \mu \end{cases}$$

$\mu < 0:$ $x = 0$ is stable, $x = \mu$ is unstable

$\mu > 0:$ $x = 0$ is unstable, $x = \mu$ is stable

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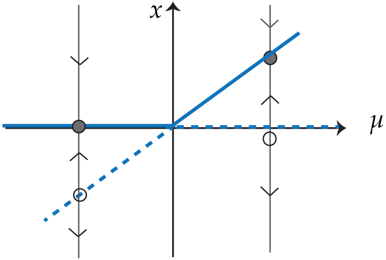

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	$\mu < 0$	$\mu > 0$
$\left.\frac{\partial f}{\partial x}\right\vert_{x=0} = \mu$	stable	unstable
$\left.\frac{\partial f}{\partial x}\right\vert_{x=\mp\sqrt{\mu}} = -2\mu$	N/A	stable

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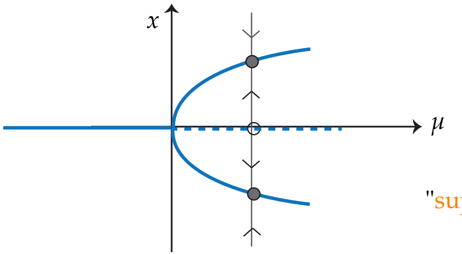

"supercritical pitchfork"

Pitchfork Bifurcation (cont.)

Example: $\dot{x} = \mu x + x^3$

Pitchfork Bifurcation (cont.)

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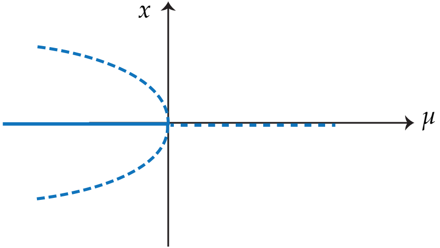


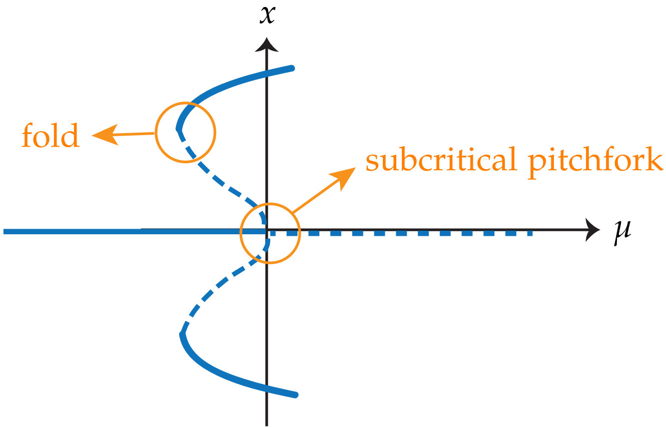
"subcritical pitchfork"

Pitchfork Bifurcation (cont.)

Example: $\dot{x} = \mu x + x^3 - x^5$

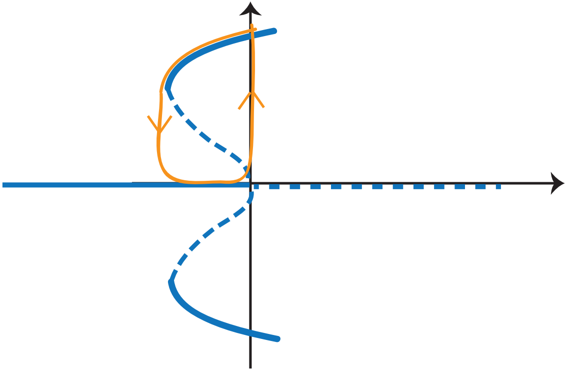
Pitchfork Bifurcation (cont.)

Example: $\dot{x} = \mu x + x^3 - x^5$

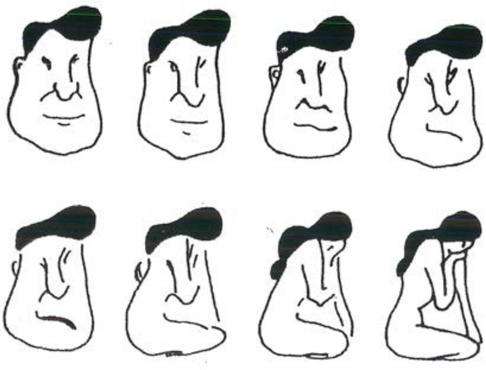


Pitchfork Bifurcation (cont.)

Hysteresis arising from a subcritical pitchfork bifurcation:



Bifurcation and hysteresis in perception



Observe the transition from a man's face to a sitting woman as you trace the figures from left to right, starting with the top row. When does the opposite transition happen as you trace back from the end to the beginning? [Fisher, 1967]

Higher Order Systems

- Fold, transcritical, and pitchfork are one-dimensional bifurcations, as evident from the first order examples above.
- They occur in higher order systems too, but are restricted to a one-dimensional *manifold.*

1D subspace: $c_1^T x = \cdots = c_{n-1}^T x = 0$

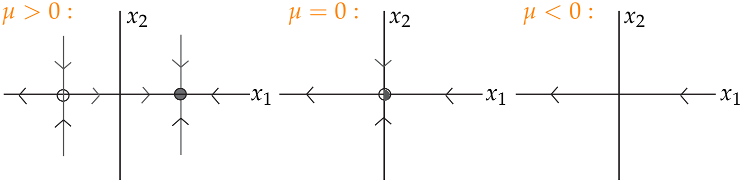
1D manifold: $g_1(x) = \cdots = g_{n-1}(x) = 0$

Higher Order Systems: Example

Example 1:

$$\dot{x}_1 = \mu - x_1^2$$
$$\dot{x}_2 = -x_2$$

A fold bifurcation occurs on the invariant $x_2 = 0$ subspace:

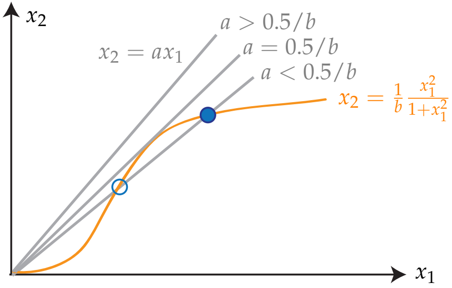


Higher Order Systems: Example

Example 2: bistable switch (Lecture 1)

$$\dot{x}_1 = -ax_1 + x_2$$
$$\dot{x}_2 = \frac{x_1^2}{1+x_1^2} - bx_2$$

A fold bifurcation occurs at $\mu \triangleq ab = 0.5$:



One-Dimensional Bifurcations

Characteristic of one-dimensional bifurcations:

$$\left.\frac{\partial f}{\partial x}\right|_{\mu=\mu^c, x=x^*(\mu^c)} \quad \text{has an eigenvalue at zero}$$

where $x^*(\mu)$ is the equilibrium point undergoing bifurcation and μ^c is the critical value at which the bifurcation occurs.

- Example 1:

$$\dot{x}_1 = \mu - x_1^2$$
$$\dot{x}_2 = -x_2$$

- Example 2:

$$\dot{x}_1 = -ax_1 + x_2$$
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Example 1 previously:

$$\left.\frac{\partial f}{\partial x}\right|_{\mu=0, x=0} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow \lambda_{1,2} = \boxed{0}, -1$$

Example 2 previously:

$$\left.\frac{\partial f}{\partial x}\right|_{\mu=\frac{1}{2}, x_1=1, x_2=a} = \begin{bmatrix} -a & 1 \\ \frac{1}{2} & -b \end{bmatrix} \rightarrow \lambda_{1,2} = \boxed{0}, -(a+b)$$

- Example 1:

$$\dot{x}_1 = \mu - x_1^2$$
$$\dot{x}_2 = -x_2$$

- Example 2:

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Hopf Bifurcation

Two-dimensional bifurcation unlike the one-dimensional types previously.

Example: Supercritical Hopf bifurcation

$$\begin{aligned}\dot{x}_1 &= x_1(\mu - x_1^2 - x_2^2) - x_2 \\ \dot{x}_2 &= x_2(\mu - x_1^2 - x_2^2) + x_1\end{aligned}$$

In polar coordinates:

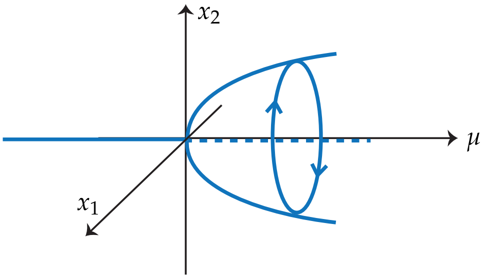
$$\begin{aligned}\dot{r} &= \mu r - r^3 \\ \dot{\theta} &= 1\end{aligned}$$

Note that a positive equilibrium for the r subsystem means a limit cycle in the (x_1, x_2) plane.

Hopf Bifurcation (Example cont.)

$\mu < 0$: stable equilibrium at $r = 0$

$\mu > 0$: unstable equilibrium at $r = 0$ and stable limit cycle at $r = \sqrt{\mu}$



The origin loses stability at $\mu = 0$ and a stable limit cycle emerges.

Hopf Bifurcation (Example 2)

Example: Subcritical Hopf bifurcation

$$\begin{aligned} \dot{r} &= \mu r + r^3 - r^5 \\ \dot{\theta} &= 1 \end{aligned}$$

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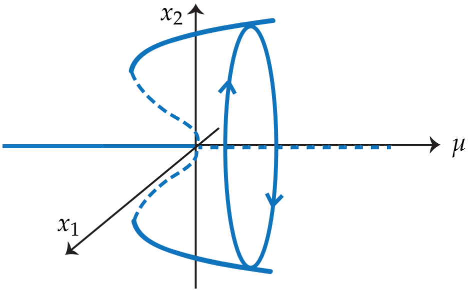
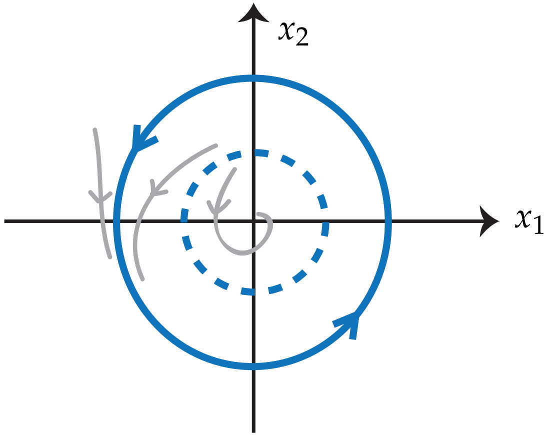


Phase portrait for $-0.25 < \mu < 0$:



Characteristic of the Hopf Bifurcation

Characteristic of the Hopf bifurcation:

$$\left.\frac{\partial f}{\partial x}\right|_{\mu=\mu^c,\, x=x^*(\mu^c)} \text{ has complex conjugate eigenvalues on the imaginary axis.}$$