Lecture 4 – ME6402, Spring 2025
Periodic Orbits in the Plane Continued

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Goals of Lecture 4

- Bendixon's Theorem (recall from last lecture)
- Poincaré-Bendixson Theorem
- Index Theory

Additional Reading

- Khalil, Chapter 2.6

Periodic Orbits in the Plane: Existence and Absence

Two criteria:

1. Bendixson (absence of periodic orbits)
2. Poincaré-Bendixson (existence of periodic orbits)

Bendixson's Theorem (from last lecture): For a time-invariant planar system

$$\dot{x}_1 = f_1(x_1, x_2) \quad \dot{x}_2 = f_2(x_1, x_2),$$

if $\nabla \cdot f(x) = \dfrac{\partial f_1}{\partial x_1} + \dfrac{\partial f_2}{\partial x_2}$ is not identically zero and does not change sign in a simply connected region D, then there are no periodic orbits lying entirely in D.

- *Compact* means closed and bounded

Periodic Orbits in the Plane: Existence and Absence

Two criteria:

1. Bendixson (absence of periodic orbits)
2. Poincaré-Bendixson (existence of periodic orbits)

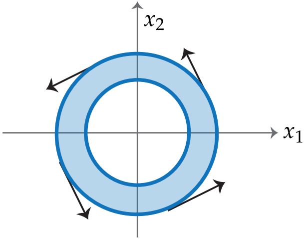
Poincaré-Bendixson Theorem: Suppose M is compact and positively invariant for the planar, time invariant system $\dot{x} = f(x), x \in \mathbb{R}^2$. If M contains no equilibrium points, then it contains a periodic orbit.

- *Compact* means closed and bounded

Example

Example: Harmonic Oscillator

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1. \end{aligned}$$



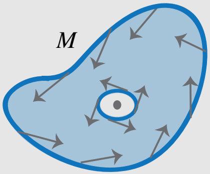
For any $R > r > 0$, the ring $\{x : r^2 \leq x_1^2 + x_2^2 \leq R^2\}$ is compact, invariant and contains no equilibria $\Rightarrow$ at least one periodic orbit. (We know there are infinitely many in this case.)

Including an Unstable Equilibrium

The "no equilibrium" condition in the PB theorem can be relaxed as:

"If M contains one equilibrium which is an unstable focus or unstable node"

Proof sketch: Since the equilibrium is an unstable focus or node, we can encircle it with a small closed curve on which $f(x)$ points outward. Then the set obtained from M by carving out the interior of the closed curve is positively invariant and contains no equilibrium.



Example

Example 2, Lecture 3:

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= -2x_1 + x_2 - x_2(x_1^2 + x_2^2)\end{aligned}$$

B_r is positively invariant for $r \geq \sqrt{\frac{3}{2}}$ but contains the equilibrium $x = 0$.

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} =$$

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B_r is positively invariant for $r \geq \sqrt{\frac{3}{2}}$ but contains the equilibrium $x = 0$.

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \qquad \lambda_{1,2} = 1 \mp j\sqrt{2} \qquad \text{unstable focus.}$$

Therefore, B_r must contain a periodic orbit.

Further Comments on Poincaré-Bendixson

A more general form of the PB Theorem states that, for time invariant, planar systems, bounded trajectories converge to equilibria, periodic orbits, or unions of equilibria connected by trajectories.

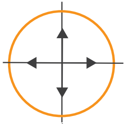
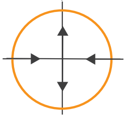
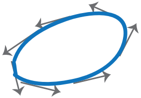
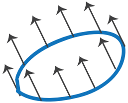
Corollary: No chaos for time invariant planar systems.

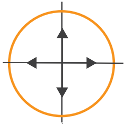
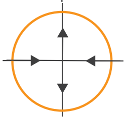
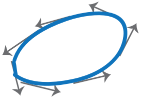
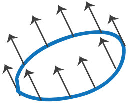
Index Theory

Again, applicable only to planar systems.

Definition (index): The index of a closed curve is k if, when traversing the curve in one direction, $f(x)$ rotates by $2\pi k$ in the same direction. The index of an equilibrium is defined to be the index of a small curve around it that doesn't enclose another equilibrium.

Index Theory: Table of Indices

type of equilibrium or curve	index	
node, focus, center	?	
saddle	?	
any closed orbit	?	
a closed curve not encircling any equilibria	?	

type of equilibrium or curve	index	
node, focus, center	+1	
saddle	-1	
any closed orbit	+1	
a closed curve not encircling any equilibria	0	

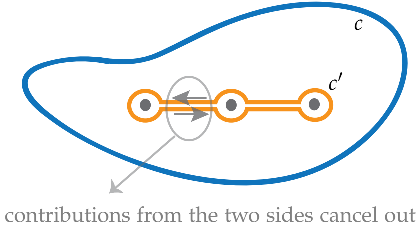
Index Theory: Closed Curve Encircling No Equilibria

The last claim (index $= 0$) follows from the following observations:

- Continuously deforming a closed curve without crossing equilibria leaves its index unchanged.
- A curve not encircling equilibria can be shrunk to an arbitrarily small one, so $f(x)$ can be considered constant.

Theorem: The index of a closed curve is equal to the sum of indices of the equilibria inside.

Graphical proof: Shrinking curve c to c' below without crossing equilibria does not change the index. The index of c' is the sum of the indices of the curves encircling the equilibria because the thin "pipes" connecting these curves do not affect the index of c'.



A Corollary for Absence of Periodic Orbits

The following corollary is useful for ruling out periodic orbits (like Bendixson's Theorem studied in the previous lecture):

Corollary: Inside any periodic orbit there must be at least one equilibrium and the indices of the equilibria enclosed must add up to $+1$.

Example

Example (from last lecture):

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\delta x_2 + x_1 - x_1^3 + x_1^2 x_2 \quad \delta > 0\end{aligned}$$

Bendixson's Criterion: No periodic orbit can lie entirely in one of the regions $x_1 \leq -\sqrt{\delta}$, $-\sqrt{\delta} \leq x_1 \leq \sqrt{\delta}$, or $x_1 \geq \sqrt{\delta}$.

Example (cont.)

Now apply the corollary above.

Equilibria: $(0,0)$, $(\mp 1,0)$. To find their indices evaluate the Jacobian:

$$\left.\frac{\partial f}{\partial x}\right|_{x=(0,0)} =$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\delta x_2 + x_1 - x_1^3 + x_1^2 x_2 \end{aligned}$$

Example (cont.)

Now apply the corollary above.

Equilibria: $(0,0)$, $(\mp 1,0)$. To find their indices evaluate the Jacobian:

$$\left.\frac{\partial f}{\partial x}\right|_{x=(0,0)} = \begin{bmatrix} 0 & 1 \\ 1 & -\delta \end{bmatrix} \quad \lambda^2 + \delta\lambda \underbrace{-1}_{<0} = 0.$$

The eigenvalues are real and have opposite signs, therefore $(0,0)$ is a saddle: index $= -1$.

$$\left.\frac{\partial f}{\partial x}\right|_{x=(\mp 1,0)} =$$

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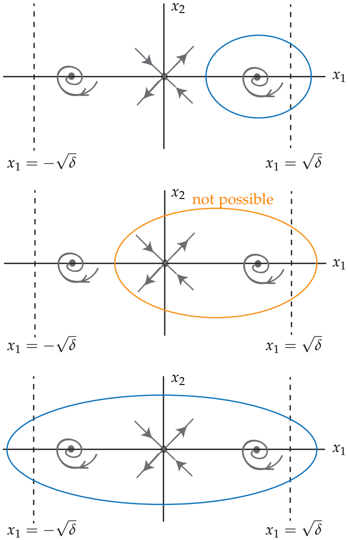
$$\left.\frac{\partial f}{\partial x}\right|_{x=(\mp 1,0)} = \begin{bmatrix} 0 & 1 \\ -2 & 1-\delta \end{bmatrix} \quad \lambda^2 + (\delta - 1)\lambda \underbrace{+2}_{>0} = 0.$$

The eigenvalues are either real with the same sign (node) or complex conjugates (focus or center), therefore $(\mp 1, 0)$ each has index$= +1$.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\delta x_2 + x_1 - x_1^3 + x_1^2 x_2 \end{aligned}$$

Example (cont.)



- Thus, the corollary above rules out the periodic orbit in the middle plot. It does not rule out the others, but does not prove their existence either. Bendixson's Criterion rules out none of the three.