Lecture 3 – ME6402, Spring 2025

Phase Portraits of Nonlinear Systems Near Hyperbolic Equilibria

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Goals of Lecture 3

- Hartman-Grobman Theorem
- Bendixson's Theorem
- Invariant Sets

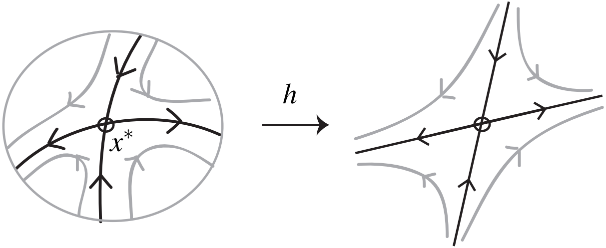
Additional Reading

- Khalil, Chapter 2

Hyperbolic equilibrium

hyperbolic equilibrium: linearization has no eigenvalues on the imaginary axis

Phase portraits of nonlinear systems near hyperbolic equilibria are qualitatively similar to the phase portraits of their linearization. According to the Hartman-Grobman Theorem (coming up) a "continuous deformation" maps one phase portrait to the other.



Hartman-Grobman Theorem

Hartman-Grobman Theorem: If x^* is a hyperbolic equilibrium of $\dot{x} = f(x), x \in \mathbb{R}^n$, then there exists a *homeomorphism* $z = h(x)$ defined in a neighborhood of x^* that maps trajectories of $\dot{x} = f(x)$ to those of $\dot{z} = Az$ where $A \triangleq \left. \frac{\partial f}{\partial x} \right|_{x=x^*}$.

- A homeomorphism is a continuous map with a continuous inverse

Hartman-Grobman Theorem: A non-example

The hyperbolicity condition can't be removed:

Example:

$$\begin{aligned} \dot{x}_1 &= -x_2 + a x_1 (x_1^2 + x_2^2) \\ \dot{x}_2 &= x_1 + a x_2 (x_1^2 + x_2^2) \end{aligned} \quad \Longrightarrow$$

Hartman-Grobman Theorem: A non-example

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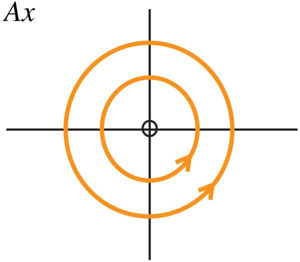
$$\begin{aligned} \dot{x}_1 &= -x_2 + a x_1 (x_1^2 + x_2^2) \\ \dot{x}_2 &= x_1 + a x_2 (x_1^2 + x_2^2) \end{aligned} \implies \begin{aligned} \dot{r} &= a r^3 \\ \dot{\theta} &= 1 \end{aligned}$$

$$x^* = (0,0) \qquad A = \left.\frac{\partial f}{\partial x}\right|_{x=x^*} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

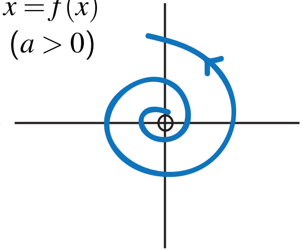
Hartman-Grobman Theorem: A non-example (cont.)

There is no continuous deformation that maps the phase portrait of the linearization to that of the original nonlinear model:

$\dot{x} = Ax$



$\dot{x} = f(x)$
$(a > 0)$



- Nonlinear model, polar coordinates:

$$\dot{r} = ar^3$$
$$\dot{\theta} = 1$$

- Linearization: $\dot{x} = Ax$,

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Periodic Orbits in the Plane

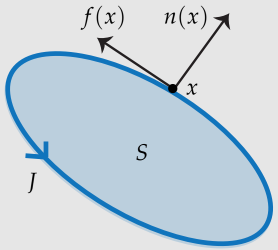
Bendixson's Theorem: For a time-invariant planar system

$$\dot{x}_1 = f_1(x_1, x_2) \quad \dot{x}_2 = f_2(x_1, x_2),$$

if $\nabla \cdot f(x) = \dfrac{\partial f_1}{\partial x_1} + \dfrac{\partial f_2}{\partial x_2}$ is not identically zero and does not change sign in a simply connected region D, then there are no periodic orbits lying entirely in D.

Proof: By contradiction. Suppose a periodic orbit J lies in D. Let S denote the region enclosed by J and $n(x)$ the normal vector to J at x. Then $f(x) \cdot n(x) = 0$ for all $x \in J$. By the Divergence Theorem:

$$\underbrace{\int_J f(x) \cdot n(x) d\ell}_{=0} = \underbrace{\iint_S \nabla \cdot f(x) dx}_{\neq 0}.$$



Example

Example: $\dot{x} = Ax, x \in \mathbb{R}^2$ can have periodic orbits only if $\text{Trace}(A) = 0$, *e.g.*,

$$A = \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix}.$$

Example

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$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\delta x_2 + x_1 - x_1^3 + x_1^2 x_2 \quad \delta > 0\end{aligned}$$

Then

$$\nabla \cdot f(x) =$$

Example

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$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\delta x_2 + x_1 - x_1^3 + x_1^2 x_2 \quad \delta > 0\end{aligned}$$

Then

$$\nabla \cdot f(x) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = x_1^2 - \delta$$

Therefore, no periodic orbit can lie entirely in the region $x_1 \leq -\sqrt{\delta}$ where $\nabla \cdot f(x) \geq 0$, or $-\sqrt{\delta} \leq x_1 \leq \sqrt{\delta}$ where $\nabla \cdot f(x) \leq 0$, or $x_1 \geq \sqrt{\delta}$ where $\nabla \cdot f(x) \geq 0$.

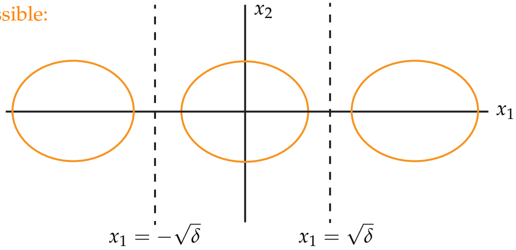
Example (cont.)

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\delta x_2 + x_1 - x_1^3 + x_1^2 x_2$$

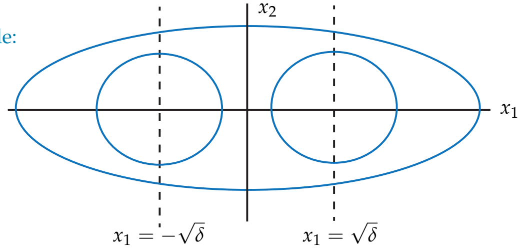
- No periodic orbit can lie entirely in the regions:
  - $x_1 \leq -\sqrt{\delta}$
  - $-\sqrt{\delta} \leq x_1 \leq \sqrt{\delta}$
  - $x_1 \geq \sqrt{\delta}$

Example (cont.)

not possible:



possible:



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Invariant Sets

Notation: $\phi(t, x_0)$ denotes a trajectory of $\dot{x} = f(x)$ with initial condition $x(0) = x_0$.

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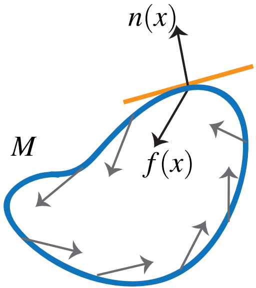
Definition: A set $M \subset \mathbb{R}^n$ is positively (negatively) invariant if, for each $x_0 \in M$, $\phi(t, x_0) \in M$ for all $t \geq 0$ $(t \leq 0)$.



If $f(x) \cdot n(x) \leq 0$ on the boundary then M is positively invariant.

Example

Example 1: A predator-prey model

$$\dot{x} = (a - by)x \qquad \text{Prey (exponential growth when } y = 0)$$

$$\dot{y} = (cx - d)y \qquad \text{Predator (exponential decay when } x = 0)$$

$$a, b, c, d, > 0$$

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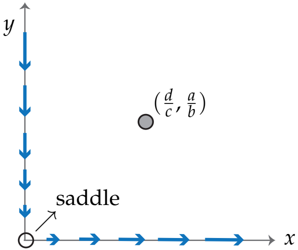
The nonnegative quadrant is invariant:



Example 2

Example 2:

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2 - x_1(x_1^2 + x_2^2)\\ \dot{x}_2 &= -2x_1 + x_2 - x_2(x_1^2 + x_2^2)\end{aligned}$$

Show that $B_r \triangleq \{x | x_1^2 + x_2^2 \leq r^2\}$ is positively invariant for sufficiently large r.

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$$\dot{x}_2 = -2x_1 + x_2 - x_2(x_1^2 + x_2^2)$$

Show that $B_r \triangleq \{x | x_1^2 + x_2^2 \leq r^2\}$ is positively invariant for sufficiently large r.

$$f(x) \cdot n(x) = x_1^2 + x_1x_2 - x_1^2(x_1^2 + x_2^2) - 2x_1x_2 + x_2^2 - x_2^2(x_1^2 + x_2^2)$$
$$= -x_1x_2 + (x_1^2 + x_2^2) - (x_1^2 + x_2^2)^2$$
$$-x_1x_2 \leq \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \quad \text{(completion of squares)}$$

Therefore, $f(x) \cdot n(x) \leq \frac{3}{2}r^2 - r^4 \leq 0$ if $r^2 \geq \frac{3}{2}$.