

Lecture 26 – ME6402, Spring 2025

Hybrid Systems and Zero Dynamics

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Goals of Lecture 26

- ▶ Introduce Hybrid Zero Dynamics
- ▶ Model walking as a hybrid system

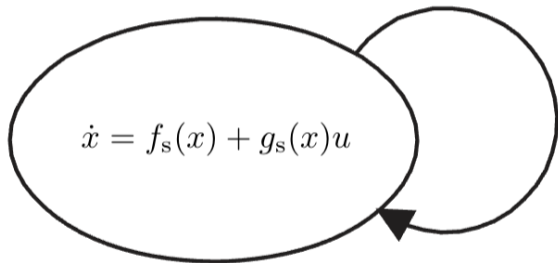
Additional Reading

- ▶ E. R. Westervelt, J. W. Grizzle, C. Chevallereau, J.-H. Choi, and B. Morris, "Feedback Control of Dynamic Bipedal Robot Locomotion". 2007

Hybrid Systems

Definition: A (simple) *hybrid control system* is a tuple

$$\mathcal{HC} = (\mathcal{D}, U, S, \Delta, f, g)$$



$$x^+ = \Delta(x^-)$$

Invariance from CBF: Theorem

Theorem: If h is a control barrier function for (1), then the following hold:

- 1 $U(x) \neq \emptyset$ for all x ;
- 2 Any Lipschitz feedback control $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$ satisfying $u(x) \in U(x)$ renders \mathcal{C} invariant;
- 3 A feedback control is given by

$$u^*(x) = \begin{cases} 0 & \text{if } \nabla h(x)^T f(x) + \alpha(h(x)) \geq 0 \\ \frac{-\nabla h(x)^T f(x) - \alpha(h(x))}{\|\nabla h(x)^T g(x)\|_2^2} (g(x)^T \nabla h(x)) & \\ \text{otherwise.} & \end{cases}$$

A sufficient condition for $u^*(x)$ to be Lipschitz on some domain is that $\nabla h(x)^T g(x) \neq 0$ everywhere on the domain.

- ▶ Control-affine system

$$\dot{x} = f(x) + g(x)u \quad (1)$$

$U(x) =$

$$\{u \in \mathbb{R}^m \mid \nabla h(x)^T (f(x) + g(x)u) \geq -\alpha(h(x))\}.$$

Example 1: Cart-Pole System Revisited

Example: Recall the model of the cart-pole system:

$$\ddot{y} = \dot{v} = \frac{1}{1 + \sin^2 \theta} \left(u + \dot{\theta}^2 \sin \theta - g \sin \theta \cos \theta \right)$$
$$\ddot{\theta} = \frac{1}{1 + \sin^2 \theta} \left(-u \cos \theta - \dot{\theta}^2 \cos \theta \sin \theta + 2g \sin \theta \right)$$

Unlike last lecture, suppose we want y to satisfy $-L \leq y \leq L$. Try

$$h(x) = \frac{1}{2}(-y^2 + L^2)$$

$$\alpha(s) = \gamma s, \quad \gamma > 0.$$

But $\nabla h(x)^T g(x) \equiv 0$. Then h cannot be a CBF because the control input vanishes from (2).

► CBF condition:

$$\sup_{u \in \mathbb{R}^m} \nabla h(x)^T (f(x) + g(x)u) \geq -\alpha(h(x)) \quad (2)$$

Control Barrier Functions for Systems with Relative Degree > 1

For systems such that $\dot{h}(x)$ does not depend on u , we need h that depends on more state variables. There is a systematic way to do this. Suppose h satisfies $\nabla h(x)^T g(x) \equiv 0$ and cannot be used as a CBF. Define

$$\Psi_1(x) = \nabla h(x)^T f(x) + \alpha_1(h(x))$$

for some Lipschitz α_1 satisfying $\alpha_1(0) = 0$, and let

$$\mathcal{C}_1 = \{x \mid \Psi_1(x) \geq 0\}.$$

- ▶ Recall our definition of relative degree. If $\dot{h}(x)$ does not depend on u , then the relative degree of the system is > 1

Control Barrier Functions for Systems with Relative Degree > 1

Lemma: Suppose $u(x)$ is a feedback control such that \mathcal{C}_1 is invariant. Then $\mathcal{C} \cap \mathcal{C}_1$ is also invariant, where $\mathcal{C} = \{x : h(x) \geq 0\}$.

Proof. Consider $x_0 \in \mathcal{C} \cap \mathcal{C}_1$ and let $x(t)$ be corresponding closed-loop trajectory. Then $x(t) \in \mathcal{C}_1$ for all $t \geq 0$ by assumption, and therefore

$$\dot{h}(x(t)) = \nabla h(x(t))^T f(x(t)) \geq -\alpha_1(h(x(t))).$$

Since $h(x_0) \geq 0$ by assumption, $h(x(t)) \geq 0$ for all $t \geq 0$ by the Comparison Lemma. \square

- ▶ $\Psi_1(x) = \nabla h(x)^T f(x) + \alpha_1(h(x))$
- ▶ $\mathcal{C}_1 = \{x \mid \Psi_1(x) \geq 0\}$.

Control Barrier Functions for Systems with Relative Degree > 1

How to ensure \mathcal{C}_1 is invariant? Use $\Psi_1(x)$ as a CBF!

- ▶ If $\nabla\Psi_1(x)^T g(x) \equiv 0$, repeat the process, defining $\Psi_2(x) = \nabla\Psi_1(x)^T f(x) + \alpha_2(\Psi_1(x))$.
- ▶ $h(x)$ is called a *high-order CBF of degree r* when this process ends with a CBF $\Psi_r(x)$.

How many times will we repeat, *i.e.*, what is r ? This is related to relative degree (next slide)

- ▶ $\Psi_1(x) = \nabla h(x)^T f(x) + \alpha_1(h(x))$
- ▶ $\mathcal{C}_1 = \{x \mid \Psi_1(x) \geq 0\}$.

High-order CBFs and Relative Degree

- ▶ *Least relative degree* r is the minimum relative degree over all states x . Therefore, $L_g L_f^{r-1} h(x) \neq 0$ for some x , but not necessarily all x .

For the previous construction to lead to valid CBF, we need:
 $L_f \Psi_r(x) + \alpha_r(\Psi_r(x)) \geq 0$ whenever $L_g \Psi_r(x) = 0$.

- ▶ Note that $L_g \Psi_r(x) = L_g L_f^{r-1} h(x)$. Therefore, states such that $L_g L_f^{r-1} h(x) = 0$ become important (more on this later).

- ▶ With h interpreted as output, recall that the system has relative degree r at x when $L_g L_f^{r-1} h(x) \neq 0$ in a neighborhood of x .

Example 2

Example. Consider the double integrator $\ddot{x}_1 = u$, (i.e., $\dot{x}_1 = x_2$, $\dot{x}_2 = u$) and suppose we want $x_1 \leq L$ always. Choose

$$h(x) = L - x_1$$

Then $\nabla h(x)^T g(x) \equiv 0$. Choose $\alpha_1(s) = \gamma_1 s$ and let

$$\Psi_1(x) = \nabla h(x)^T f(x) + \alpha_1(h(x)) = -x_2 + \gamma_1(L - x_1).$$

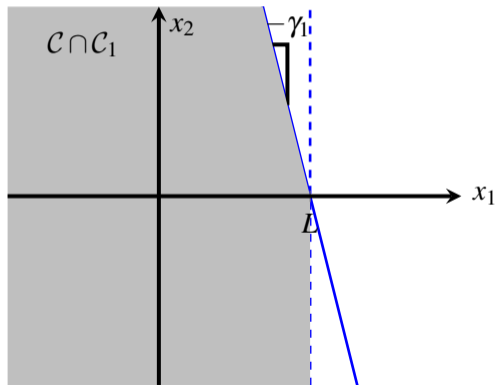
Then

$$\nabla \Psi_1(x)^T g(x) = -1$$

and we can use $\Psi_1(x)$ as a valid CBF.

Example 2 (cont.)

- ▶ $\mathcal{C} = \{x \mid h(x) \geq 0\} = \{x \mid x_1 \leq L\}$
- ▶ $\mathcal{C}_1 = \{x \mid \Psi_1(x) \geq 0\} = \{x \mid \gamma_1(L - x_1) \geq x_2\}$.



- ▶ $\ddot{x} = u$

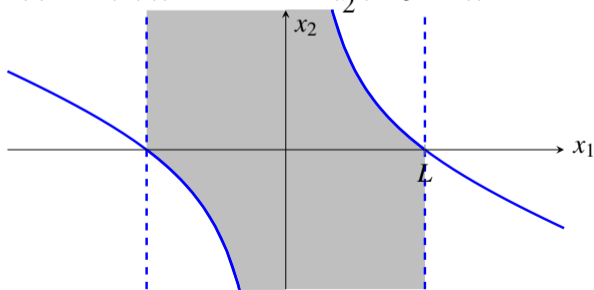
Example 3

Let's try $\ddot{x}_1 = u$ again, but with safe set $-L \leq x \leq L$. Choose $h(x) = \frac{1}{2}(-x_1^2 + L^2)$. Then

$$\dot{h}(x) = -x_1x_2, \quad \ddot{h}(x) = -x_2^2 - x_1u$$

Construct

$$\Psi(x) = \dot{h}(x) + \alpha_1(h(x)) = -x_1x_2 + \alpha_1\left(\frac{1}{2}(-x_1^2 + L^2)\right)$$



Example 3

Let's try $\ddot{x}_1 = u$ again, but with safe set $-L \leq x \leq L$. Choose $h(x) = \frac{1}{2}(-x_1^2 + L^2)$. Then

$$\dot{h}(x) = -x_1x_2, \quad \ddot{h}(x) = -x_2^2 - x_1u$$

Construct

$$\Psi(x) = \dot{h}(x) + \alpha_1(h(x)) = -x_1x_2 + \alpha_1\left(\frac{1}{2}(-x_1^2 + L^2)\right)$$

$$\dot{\Psi}(x) = \ddot{h}(x) + \alpha_1'(h(x))\dot{h}(x) = -x_2^2 + \underbrace{x_1}_{L_g L_f h} u + \alpha_1'\left(\frac{1}{2}(-x_1^2 + L^2)\right)(-x_1x_2)$$

- ▶ Least Relative degree is $r = 2$.
- ▶ Note $L_g \Psi(x) = L_g L_f h(x)$. Is it possible that $L_g L_f h(x) = 0$? Yes! Whenever $x_1 = 0$.
- ▶ Is this a problem? Need to investigate further...

Example 3

- ▶ We need to see if we can find α_2 such that $\dot{\Psi}(x) + \alpha_2(\Psi(x)) \geq 0$ whenever $x_1 = 0$, $\Psi(x) \geq 0$, and $h(x) \geq 0$:

$$\dot{\Psi}(x) + \alpha_2(\Psi(x))|_{x_1=0} = -x_2^2 + \alpha_2(\alpha_1(L^2/2))$$

Always possible to find x_2 large enough so that $-x_2^2 + \alpha_2(\alpha_1(L^2/2)) < 0$, regardless of α_1 and α_2 , so Ψ is not a valid CBF. What to do?

- ▶ Option 1: Nothing, except make sure α_1 and α_2 have sufficient slope so that this is only a problem when x_2 is very large \implies practical, but lose theoretical guarantees
- ▶ Option 2: try a different high-order CBF h (next slide)

$$\begin{aligned}\Psi(x) &= -x_1x_2 + \alpha_1\left(\frac{1}{2}(-x_1^2 + L^2)\right) \\ \dot{\Psi}(x) &= -x_2^2 + x_1u \\ &\quad + \alpha_1'\left(\frac{1}{2}(-x_1^2 + L^2)\right)(-x_1x_2)\end{aligned}$$

Example 4

Same system: $\ddot{x}_1 = u$. Try

$$h(x) = \frac{1}{4}(-x_1^4 + L^4), \quad \dot{h}(x) = -x_1^3 x_2, \quad \ddot{h}(x) = 3x_1^2 x_2^2 + x^3 u$$

Let

$$\Psi(x) = \dot{h} + \gamma_1 h = -x_1^3 x_2 + \frac{\gamma_1}{4}(-x_1^4 + L^4)$$

$$\dot{\Psi}(x) = \ddot{h} + \gamma_1 \dot{h} = 3x_1^2 x_2^2 - x_1^3 u - \gamma_1 x_1^3 x_2$$

Then, still, $x_1^3 = 0$ whenever $x_1 = 0$. But

$$L_f \Psi(x) = 3x_1^2 x_2^2 - \gamma_1 x_1^3 x_2$$

and therefore $L_f \Psi = 0$ whenever $L_g \Psi = 0$. This means Ψ satisfies the CBF constraint $\sup_u L_f \Psi(x) + L_g \Psi(x)u \geq -\alpha_2(\Psi(x))$ for any α_2 , and Ψ is a valid CBF.

Accommodating Loss of Relative Degree

- ▶ What was the key in the previous example? A: Make sure $L_f \Psi = 0$ whenever $L_g \Psi = 0$.

A systematic method for system with least relative degree r :

- 1 Let $\mathcal{S} = \{x \mid L_g L_f^{r-1} h(x) = 0\}$ (Recall that $L_g \Psi = L_g L_f^{r-1} h$)
- 2 If \mathcal{S} is in the interior of $\mathcal{C} = \{h(x) \geq 0\}$, find $\varepsilon > 0$ such that $\mathcal{S} \subseteq \{h(x) \geq \varepsilon\}$. (Else: method does not work)
- 3 Construct new $\tilde{h}(x)$ that saturates to 1 as $h(x) \rightarrow \varepsilon$: Let $\tilde{h}(x) = \sigma(h(x)/\varepsilon)$ where σ is any r -differentiable function satisfying:

$$\begin{cases} \sigma(0) = 0 \\ \sigma(\tau) = 1 \text{ for all } \tau \geq 1 \\ \sigma'(\tau) > 0 \text{ for all } \tau < 1 \end{cases}$$

- ▶ Tan, Cortez, Dimarogonas, "High-order Barrier Functions: Robustness, Safety and Performance-Critical Control", arXiv:2104.00101

Accommodating Loss of Relative Degree

- ▶ For homework and project, if needed, can take Option 1.

Demo: Cart-Pole System with Restriction on θ

Try $h(x) = L - \theta$.

$$\ddot{y} = \frac{1}{1 + \sin^2 \theta} \left(u + \dot{\theta}^2 \sin \theta - g \sin \theta \cos \theta \right)$$
$$\ddot{\theta} = \frac{1}{1 + \sin^2 \theta} \left(-u \cos \theta - \dot{\theta}^2 \cos \theta \sin \theta + 2g \sin \theta \right)$$