Lecture 25 – ME6402, Spring 2025

High-Order Control Barrier Functions

Maegan Tucker

April 10, 2025

Goals of Lecture 25

- Extend Control Barrier Functions to systems with relative degree > 1

Control Barrier Functions

Definition: A function h with $\mathcal{C} = \{x \mid h(x) \geq 0\}$ is a *control barrier function (CBF)* for (2) if there exists a locally Lipschitz function $\alpha : \mathbb{R} \to \mathbb{R}$ satisfying $\alpha(0) = 0$ such that

$$\sup_{u \in \mathbb{R}^m} \nabla h(x)^T (f(x) + g(x)u) \geq -\alpha(h(x)) \quad \text{for all } x \in \mathbb{R}^n. \qquad (1)$$

We can also write (1) using Lie derivative notation:

$$\sup_{u \in \mathbb{R}^m} L_f h(x) + L_g h(x) u \geq -\alpha(h(x))$$

Define

$$U(x) = \{u \in \mathbb{R}^m \mid \nabla h(x)^T (f(x) + g(x)u) \geq -\alpha(h(x))\}.$$

- Control-affine system

$$\dot{x} = f(x) + g(x)u \qquad (2)$$

Invariance from CBF: Theorem

Theorem: If h is a control barrier function for (3), then the following hold:

1. $U(x) \neq \emptyset$ for all x;
2. Any Lipschitz feedback control $u : \mathbb{R}^n \to \mathbb{R}^m$ satisfying $u(x) \in U(x)$ renders $\mathcal{C}$ invariant;
3. A feedback control is given by

$$u^*(x) = \begin{cases} 0 \text{ if } \nabla h(x)^T f(x) + \alpha(h(x)) \geq 0 \\ \dfrac{-\nabla h(x)^T f(x) - \alpha(h(x))}{\|\nabla h(x)^T g(x)\|_2^2} (g(x)^T \nabla h(x)) \\ \text{otherwise.} \end{cases}$$

A sufficient condition for $u^*(x)$ to be Lipschitz on some domain is that $\nabla h(x)^T g(x) \neq 0$ everywhere on the domain.

- Control-affine system

$$\dot{x} = f(x) + g(x)u \qquad (3)$$

$U(x) =$
$\{u \in \mathbb{R}^m \mid \nabla h(x)^T (f(x) + g(x)u) \geq -\alpha(h(x))\}$.

Example 1: Cart-Pole System Revisited

Example: Recall the model of the cart-pole system:

$$\ddot{y} = \dot{v} = \frac{1}{1+\sin^2\theta}\left(u + \dot{\theta}^2 \sin\theta - g\sin\theta\cos\theta\right)$$

$$\ddot{\theta} = \frac{1}{1+\sin^2\theta}\left(-u\cos\theta - \dot{\theta}^2\cos\theta\sin\theta + 2g\sin\theta\right)$$

Unlike last lecture, suppose we want y to satisfy $-L \leq y \leq L$. Try

$$h(x) = \frac{1}{2}(-y^2 + L^2)$$

$$\alpha(s) = \gamma s, \quad \gamma > 0.$$

But $\nabla h(x)^T g(x) \equiv 0$. Then h cannot be a CBF because the control input vanishes from (4).

- CBF condition:

$$\sup_{u\in\mathbb{R}^m} \nabla h(x)^T (f(x) + g(x)u) \geq -\alpha(h(x)) \tag{4}$$

Control Barrier Functions for Systems with Relative Degree > 1

For systems such that $\dot{h}(x)$ does not depend on u, we need h that depends on more state variables. There is a systematic way to do this. Suppose h satisfies $\nabla h(x)^T g(x) \equiv 0$ and cannot be used as a CBF. Define

$$\Psi_1(x) = \nabla h(x)^T f(x) + \alpha_1(h(x))$$

for some Lipschitz α_1 satisfying $\alpha(0) = 0$, and let

$$\mathcal{C}_1 = \{x \mid \Psi_1(x) \geq 0\}.$$

- Recall our definition of relative degree. If $\dot{h}(x)$ does not depend on u, then the relative degree of the system is > 1

Control Barrier Functions for Systems with Relative Degree > 1

Lemma: Suppose $u(x)$ is a feedback control such that $\mathcal{C}_1$ is invariant. Then $\mathcal{C} \cap \mathcal{C}_1$ is also invariant, where $\mathcal{C} = \{x : h(x) \geq 0\}$.

Proof. Consider $x_0 \in \mathcal{C} \cap \mathcal{C}_1$ and let $x(t)$ be corresponding closed-loop trajectory. Then $x(t) \in \mathcal{C}_1$ for all $t \geq 0$ by assumption, and therefore

$$\dot{h}(x(t)) = \nabla h(x(t))^T f(x(t)) \geq -\alpha_1(h(x(t))).$$

Since $h(x_0) \geq 0$ by assumption, $h(x(t)) \geq 0$ for all $t \geq 0$ by the Comparison Lemma. □

- $\Psi_1(x) = \nabla h(x)^T f(x) + \alpha_1(h(x))$
- $\mathcal{C}_1 = \{x \mid \Psi_1(x) \geq 0\}$.

Control Barrier Functions for Systems with Relative Degree > 1

How to ensure $\mathcal{C}_1$ is invariant? Use $\Psi_1(x)$ as a CBF!

- If $\nabla\Psi_1(x)^T g(x) \equiv 0$, repeat the process, defining $\Psi_2(x) = \nabla\Psi_1(x)^T f(x) + \alpha_2(\Psi_1(x))$.
- $h(x)$ is called a *high-order CBF of degree r* when this process ends with a CBF $\Psi_r(x)$.

How many times will we repeat, *i.e.*, what is r? This is related to relative degree (next slide)

- $\Psi_1(x) = \nabla h(x)^T f(x) + \alpha_1(h(x))$
- $\mathcal{C}_1 = \{x \mid \Psi_1(x) \geq 0\}$.

High-order CBFs and Relative Degree

- *Least relative degree* r is the minimum relative degree over all states x. Therefore, $L_gL_f^{r-1}h(x) \neq 0$ for some x, but not necessarily all x.

For the previous construction to lead to valid CBF, we need: $L_f\Psi_r(x) + \alpha_r(\Psi_r(x)) \geq 0$ whenever $L_g\Psi_r(x) = 0$.

- Note that $L_g\Psi_r(x) = L_gL_f^{r-1}h(x)$. Therefore, states such that $L_gL_f^{r-1}h(x) = 0$ become important (more on this later).

- With h interpreted as output, recall that the system has relative degree r at x when $L_gL_f^{r-1}h(x) \neq 0$ *in a neighborhood* of x.

Example 2

Example. Consider the double integrator $\ddot{x}_1 = u$, (*i.e.*, $\dot{x}_1 = x_2$, $\dot{x_2} = u$) and suppose we want $x_1 \leq L$ always. Choose

$$h(x) = L - x_1$$

Then $\nabla h(x)^T g(x) \equiv 0$. Choose $\alpha_1(s) = \gamma_1 s$ and let

$$\Psi_1(x) = \nabla h(x)^T f(x) + \alpha_1(h(x)) = -x_2 + \gamma_1(L - x_1).$$

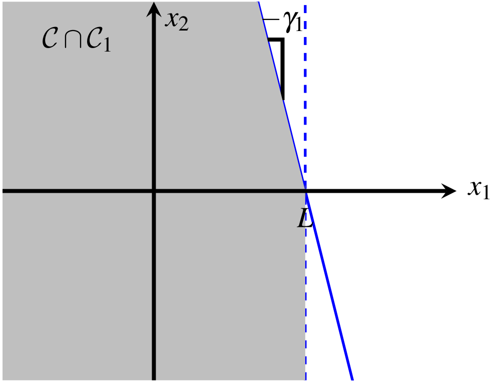
Then

$$\nabla \Psi_1(x)^T g(x) = -1$$

and we can use $\Psi_1(x)$ as a valid CBF.

Example 2 (cont.)

- $\mathcal{C} = \{x \mid h(x) \geq 0\} = \{x \mid x_1 \leq L\}$
- $\mathcal{C}_1 = \{x \mid \Psi_1(x) \geq 0\} = \{x \mid \gamma_1(L - x_1) \geq x_2\}$.



- $\ddot{x} = u$

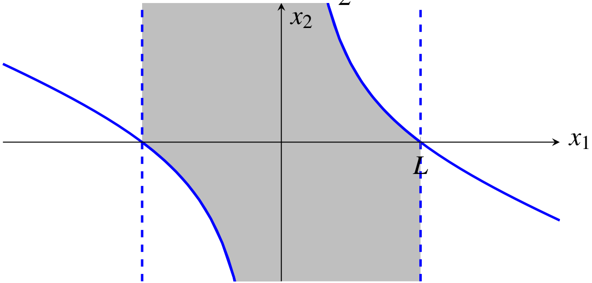
Example 3

Let's try $\ddot{x}_1 = u$ again, but with safe set $-L \leq x \leq L$. Choose $h(x) = \frac{1}{2}(-x_1^2 + L^2)$. Then

$$\dot{h}(x) = -x_1 x_2, \quad \ddot{h}(x) = -x_2^2 - x_1 u$$

Construct

$$\Psi(x) = \dot{h}(x) + \alpha_1(h(x)) = -x_1 x_2 + \alpha_1(\frac{1}{2}(-x_1^2 + L^2))$$



Example 3

Let's try $\ddot{x}_1 = u$ again, but with safe set $-L \leq x \leq L$. Choose $h(x) = \frac{1}{2}(-x_1^2 + L^2)$. Then

$$\dot{h}(x) = -x_1 x_2, \quad \ddot{h}(x) = -x_2^2 - x_1 u$$

Construct

$$\Psi(x) = \dot{h}(x) + \alpha_1(h(x)) = -x_1 x_2 + \alpha_1(\frac{1}{2}(-x_1^2 + L^2))$$

$$\dot{\Psi}(x) = \ddot{h}(x) + \alpha_1'(h(x))\dot{h}(x) = -x_2^2 + \underbrace{x_1}_{L_g L_f h} u + \alpha_1'\left(\frac{1}{2}(-x_1^2 + L^2)\right)(-x_1 x_2)$$

- Least Relative degree is $r = 2$.
- Note $L_g \Psi(x) = L_g L_f h(x)$. Is it possible that $L_g L_f h(x) = 0$? Yes! Whenever $x_1 = 0$.
- Is this a problem? Need to investigate further...

Example 3

- We need to see if we can find α_2 such that $\dot{\Psi}(x) + \alpha_2(\Psi(x)) \geq 0$ whenever $x_1 = 0$, $\Psi(x) \geq 0$, and $h(x) \geq 0$:

$$\dot{\Psi}(x) + \alpha_2(\Psi(x))\Big|_{x_1=0} = -x_2^2 + \alpha_2(\alpha_1(L^2/2))$$

Always possible to find x_2 large enough so that $-x_2^2 + \alpha_2(\alpha_1(L^2/2)) < 0$, regardless of α_1 and α_2, so Ψ is not a valid CBF. What to do?

- Option 1: Nothing, except make sure α_1 and α_2 have sufficient slope so that this is only a problem when x_2 is very large $\implies$ practical, but lose theoretical guarantees
- Option 2: try a different high-order CBF h (next slide)

$$\Psi(x) - x_1x_2 + \alpha_1(\frac{1}{2}(-x_1^2 + L^2))$$

$$\dot{\Psi}(x) = -x_2^2 + x_1u$$
$$+ \alpha_1'\left(\frac{1}{2}(-x_1^2 + L^2)\right)(-x_1x_2)$$

Example 4

Same system: $\ddot{x}_1 = u$. Try

$$h(x) = \frac{1}{4}(-x_1^4 + L^4), \quad \dot{h}(x) = -x_1^3 x_2, \quad \ddot{h}(x) = 3x_1^2 x_2^2 + x^3 u$$

Let

$$\Psi(x) = \dot{h} + \gamma_1 h = -x_1^3 x_2 + \frac{\gamma_1}{4}(-x_1^4 + L^4)$$

$$\dot{\Psi}(x) = \ddot{h} + \gamma_1 \dot{h} = 3x_1^2 x_2^2 - x_1^3 u - \gamma_1 x_1^3 x_2$$

Then, still, $x_1^3 = 0$ whenever $x_1 = 0$. But

$$L_f \Psi(x) = 3x_1^2 x_2^2 - \gamma_1 x_1^3 x_2$$

and therefore $L_f \Psi = 0$ whenever $L_g \Psi = 0$. This means Ψ satisfies the CBF constraint $\sup_u L_f \Psi(x) + L_g \Psi(x) u \geq -\alpha_2(\Psi(x))$ for any α_2, and Ψ is a valid CBF.

Accommodating Loss of Relative Degree

- What was the key in the previous example? A: Make sure $L_f\Psi = 0$ whenever $L_g\Psi = 0$.

A systematic method for system with least relative degree r:

1. Let $\mathcal{S} = \{x \mid L_g L_f^{r-1} h(x) = 0\}$ (Recall that $L_g\Psi = L_g L_f^{r-1} h$)
2. If $\mathcal{S}$ is in the interior of $\mathcal{C} = \{h(x) \geq 0\}$, find $\varepsilon > 0$ such that $\mathcal{S} \subseteq \{h(x) \geq \varepsilon\}$. (Else: method does not work)
3. Construct new $\tilde{h}(x)$ that saturates to 1 as $h(x) \to \varepsilon$: Let $\tilde{h}(x) = \sigma(h(x)/\varepsilon)$ where σ is any r-differentiable function satisfying:

$$\begin{cases} \sigma(0) = 0 \\ \sigma(\tau) = 1 \text{ for all } \tau \geq 1 \\ \sigma'(\tau) > 0 \text{ for all } \tau < 1 \end{cases}$$

- Tan, Cortez, Dimarogonas, "High-order Barrier Functions: Robustness, Safety and Performance-Critical Control", arXiv:2104.00101

Accommodating Loss of Relative Degree

- For homework and project, if needed, can take Option 1.

Demo: Cart-Pole System with Restriction on θ

Try $h(x) = L - \theta$.

$$\ddot{y} = \frac{1}{1+\sin^2\theta}\left(u + \dot{\theta}^2 \sin\theta - g\sin\theta\cos\theta\right)$$

$$\ddot{\theta} = \frac{1}{1+\sin^2\theta}\left(-u\cos\theta - \dot{\theta}^2 \cos\theta\sin\theta + 2g\sin\theta\right)$$