Lecture 24 – ME6402, Spring 2025
Control Barrier Functions

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Goals of Lecture 24

- Extend barrier functions to Control Barrier Functions

Additional Reading

- A. Ames, S. Coogan, M. Egerstedt, G. Notomista, K. Sreenath, P. Tabuada, "Control Barrier Functions: Theory and Applications," IEEE Transactions on Automatic Control, 2019

Ingenuity Flight, April 19, 2021

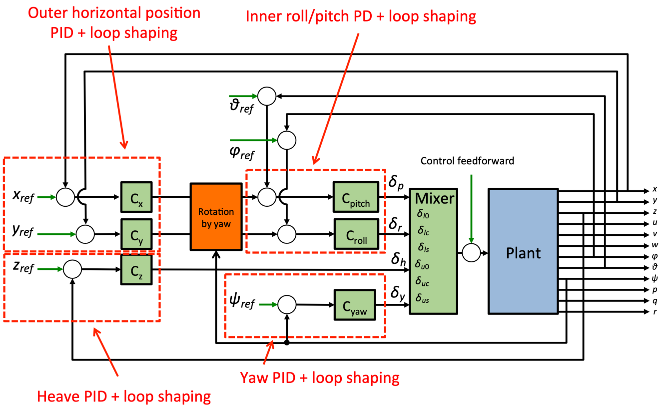


Fig. 7 **Overall control architecture. Positions in the ground frame are denoted by $[x; y; z]$; velocities in the body frame are denoted by $[u; v; w]$; Euler angles and angular rates with respect to the ground frame are denoted by $[\phi; \theta; \psi]$ and $[p; q; r]$. Commanded lower collective, lower cosine cyclic, and lower sine cyclic are denoted by $[\delta_{l0}; \delta_{lc}; \delta_{ls}]$; similar for the upper controls. The symbols δ_r, δ_p, δ_h, and δ_y denote linear combinations of the control inputs to produce inputs aligned with the roll, pitch, heave, and yaw axes.**

- Video

Control Barrier Functions

Consider a control-affine system

$$\dot{x} = f(x) + g(x)u$$

and a given set $\mathcal{C} = \{x : h(x) \geq 0\}$. How can we choose a controller $u(x)$ such that $\mathcal{C}$ is positively invariant?

Recall Barrier Functions (Last Lecture)

Definition: A function h with $\mathcal{C} = \{x \mid h(x) \geq 0\}$ is a *barrier function* for $\dot{x} = f(x)$ if there exists a locally Lipschitz function $\alpha : \mathbb{R} \to \mathbb{R}$ satisfying $\alpha(0) = 0$ such that

$$\nabla h(x)^T f(x) \geq -\alpha(h(x)) \quad \text{for } \textit{all } x \in \mathbb{R}^n.$$

Using Lie derivative notation, recall $\nabla h(x)^T f(x) = L_f h(x) = \dot{h}(x)$.

Theorem: If h is a barrier function, then $\mathcal{C} = \{x : h(x) \geq 0\}$ is positively invariant.

- In general, we think of α as being an increasing function, but this is not needed for the theory on the next slide.

Control Barrier Functions

Definition: A function h with $\mathcal{C} = \{x \mid h(x) \geq 0\}$ is a *control barrier function (CBF)* for (2) if there exists a locally Lipschitz function $\alpha : \mathbb{R} \to \mathbb{R}$ satisfying $\alpha(0) = 0$ such that

$$\sup_{u \in \mathbb{R}^m} \nabla h(x)^T (f(x) + g(x)u) \geq -\alpha(h(x)) \quad \text{for all } x \in \mathbb{R}^n. \qquad (1)$$

We can also write (1) using Lie derivative notation:

$$\sup_{u \in \mathbb{R}^m} L_f h(x) + L_g h(x)u \geq -\alpha(h(x))$$

Define

$$U(x) = \{u \in \mathbb{R}^m \mid \nabla h(x)^T (f(x) + g(x)u) \geq -\alpha(h(x))\}.$$

- Control-affine system

$$\dot{x} = f(x) + g(x)u \quad (2)$$

Invariance from CBF: Theorem

Theorem: If h is a control barrier function for (3), then the following hold:

1. $U(x) \neq \emptyset$ for all x;
2. Any Lipschitz feedback control $u : \mathbb{R}^n \to \mathbb{R}^m$ satisfying $u(x) \in U(x)$ renders $\mathcal{C}$ invariant;
3. A feedback control is given by

$$u^*(x) = \begin{cases} 0 \text{ if } \nabla h(x)^T f(x) + \alpha(h(x)) \geq 0 \\ \dfrac{-\nabla h(x)^T f(x) - \alpha(h(x))}{\|\nabla h(x)^T g(x)\|_2^2}(g(x)^T \nabla h(x)) \\ \qquad \text{otherwise.} \end{cases}$$

A sufficient condition for $u^*(x)$ to be Lipschitz on some domain is that $\nabla h(x)^T g(x) \neq 0$ everywhere on the domain.

- Control-affine system

$$\dot{x} = f(x) + g(x)u \qquad (3)$$

$U(x) =$
$\{u \in \mathbb{R}^m \mid \nabla h(x)^T (f(x) + g(x)u) \geq -\alpha(h(x))\}$.

Invariance from CBF: Proof

Proof

1. If $\sup_{u \in \mathbb{R}^m} \nabla h(x)^T (f(x) + g(x)u) < \infty$, then the sup is attained for some u.
2. h becomes a (regular) barrier function for $\tilde{f}(x) = f(x) + g(x)u(x)$ and theorem from previous lecture applies.

- CBF condition:

$$\sup_{u \in \mathbb{R}^m} \nabla h(x)^T (f(x) + g(x)u) \geq -\alpha(h(x))$$

Theorem: If h is a control barrier function, then the following hold:

1. $U(x) \neq \emptyset$ for all x;
2. Any Lipschitz feedback control $u : \mathbb{R}^n \to \mathbb{R}^m$ satisfying $u(x) \in U(x)$ renders $\mathcal{C}$ invariant;
3. A feedback control is given by

$$u^*(x) = \begin{cases} 0 \text{ if } \nabla h(x)^T f(x) + \alpha(h(x)) \geq 0 \\ \dfrac{-\nabla h(x)^T f(x) - \alpha(h(x))}{\|\nabla h(x)^T g(x)\|_2^2} (g(x)^T \nabla h(x)) \\ \qquad \text{otherwise.} \end{cases}$$

A sufficient condition for $u^*(x)$ to be Lipschitz on some domain is that $\nabla h(x)^T g(x) \neq 0$ everywhere on the domain.

Invariance from CBF: Proof

Proof

3. (Sketch) First, note that $u^*(x)$ is well-defined since $\nabla h(x)^T g(x) \neq 0$ whenever $h(x)^T f(x) + \alpha(h(x)) < 0$ by CBF condition. $u^*(x)$ can be considered as a composition of 3 Lipschitz functions and is therefore Lipschitz. Finally, it is easy to verify that

$$\nabla h(x)^T (f(x) + g(x)u^*(x) + \alpha(h(x)))$$
$$= \begin{cases} \nabla h(x)^T f(x) + \alpha(h(x)) & \text{if } \nabla h(x)^T f(x) + \alpha(h(x)) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$
$$\geq 0.$$

□

- CBF condition:

$$\sup_{u \in \mathbb{R}^m} \nabla h(x)^T (f(x) + g(x)u) \geq -\alpha(h(x))$$

Theorem: If h is a control barrier function, then the following hold:

1. $U(x) \neq \emptyset$ for all x;
2. Any Lipschitz feedback control $u : \mathbb{R}^n \to \mathbb{R}^m$ satisfying $u(x) \in U(x)$ renders $\mathcal{C}$ invariant;
3. A feedback control is given by

$$u^*(x) = \begin{cases} 0 \text{ if } \nabla h(x)^T f(x) + \alpha(h(x)) \geq 0 \\ \dfrac{-\nabla h(x)^T f(x) - \alpha(h(x))}{\|\nabla h(x)^T g(x)\|_2^2} (g(x)^T \nabla h(x)) \\ \text{otherwise.} \end{cases}$$

A sufficient condition for $u^*(x)$ to be Lipschitz on some domain is that $\nabla h(x)^T g(x) \neq 0$ everywhere on the domain.

Minimum Effort Control

<u>Remark:</u> From the above proof, specifically, the condition

$$\nabla h(x)^T (f(x) + g(x)u^*(x) + \alpha(h(x))$$
$$= \begin{cases} \nabla h(x)^T f(x) + \alpha(h(x)) & \text{if } \nabla h(x)^T f(x) + \alpha(h(x)) \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

we conclude that $u^*(x)$ is the "minimum effort" controller, *i.e.*, $u^*(x) = \text{argmin}_{u \in U(x)} \|u\|_2^2$.

Example: Cart-Pole System

Example: Recall the model of the cart-pole system from Lecture 16 (take $m = M = \ell = 1$):

$$\ddot{y} = \dot{v} = \frac{1}{1+\sin^2\theta}\left(u + \dot{\theta}^2 \sin\theta - g\sin\theta\cos\theta\right)$$

$$\ddot{\theta} = \frac{1}{1+\sin^2\theta}\left(-u\cos\theta - \dot{\theta}^2\cos\theta\sin\theta + 2g\sin\theta\right)$$

where $v = \dot{y}$ is velocity. Take as the state $x = [y \ v \ \theta \ \dot{\theta}]^T$. Suppose we want v to satisfy

$$-L \leq v \leq L.$$

Choose

$$h(x) = \frac{1}{2}(-v^2 + L^2)$$

$$\alpha(s) = \gamma s, \quad \gamma > 0.$$

Example: Cart-Pole System

Then

$$\nabla h(x)^T f(x) = L_f h(x) = \frac{-v}{1+\sin^2(\theta)}\left(\dot{\theta}^2 \sin\theta - g\sin\theta\cos\theta\right)$$

$$\nabla h(x)^T g(x) = L_g h(x) = \frac{-v}{1+\sin^2(\theta)}$$

$$\alpha(h(x)) = \gamma h(x)$$

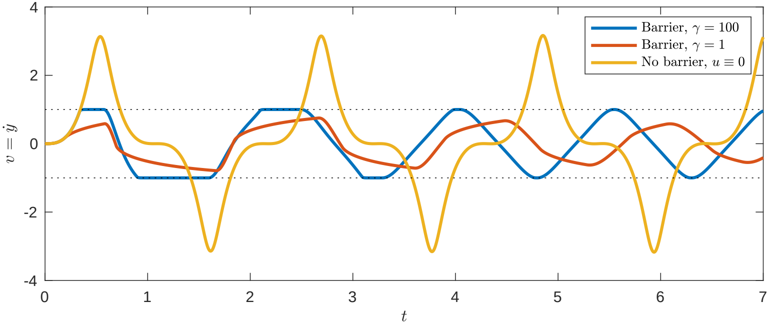
and $u^*(x)$ constructed as above.

$$\dot{v} = \frac{1}{1+\sin^2\theta}\left(u + \dot{\theta}^2 \sin\theta - g\sin\theta\cos\theta\right)$$

$$\ddot{\theta} = \frac{1}{1+\sin^2\theta}\left(-u\cos\theta - \dot{\theta}^2\cos\theta\sin\theta + 2g\sin\theta\right)$$

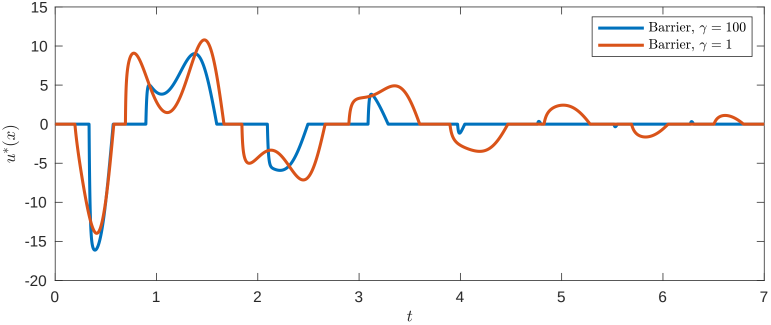
Example: Cart-Pole System

The figures below show results for $x_0 = [y_0 \;\; v_0 \;\; \theta_0 \;\; \dot{\theta}_0]^T = [0 \; 0 \; \pi/2 \; 0]^T$ using the $u^*(x)$ from the theorem.



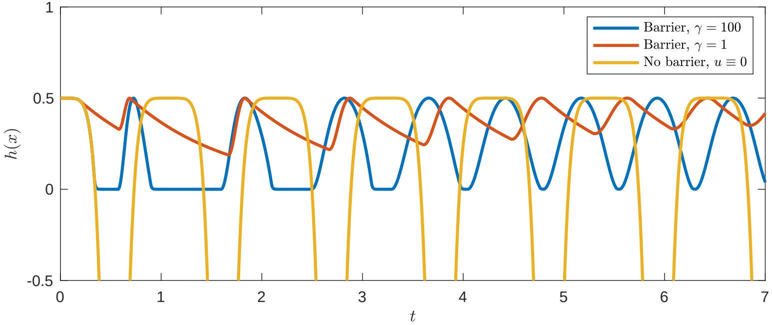
Example: Cart-Pole System

The figures below show results for $x_0 = [y_0 \; v_0 \; \theta_0 \; \dot{\theta}_0]^T = [0 \; 0 \; \pi/2 \; 0]^T$ using the $u^*(x)$ from the theorem.



Example: Cart-Pole System

The figures below show results for $x_0 = [y_0 \;\; v_0 \;\; \theta_0 \;\; \dot{\theta}_0]^T = [0\ 0\ \pi/2\ 0]^T$ using the $u^*(x)$ from the theorem.



Controller Synthesis as Optimization Problem

- $u^*(x)$ from theorem is often not the controller that's implemented.
- The original intent, and still the primary use, of CBFs is to protect other possibly unsafe controllers.

For fixed x, the CBF constraint is *affine* in u! Then we can define a *convex* program to compute a control input at each time instant:

$$u(x) = \text{arg} \min_{\mu} \quad C(\mu, x)$$
$$\text{subject to} \quad \nabla h(x)^T f(x) + \nabla h(x)^T g(x)\mu \geq -\alpha(h(x))$$

where $C(\mu, x)$ is convex in μ for each fixed x.

- CBF constraint:

$$\nabla h(x)^T (f(x) + g(x)u) \geq -\alpha(h(x))$$

Example

Example. Suppose $k(x) : \mathbb{R}^n \to \mathbb{R}^m$ is some nominal feedback controller designed for some other purpose (e.g., performance objectives). Can choose $C(\mu, x) = \|\mu - k(x)\|_2^2$. The result is a quadratic program (with affine constraints) to compute $u(x)$ at each x.

- Raises questions about solving a QP in real-time online, care must be taken with discretization values, etc.
- Convex solvers are fast enough that they can be included "in-the-loop" and have been for applications like stable bipedal locomotion, quadrotor control