# Lecture 23 – ME6402, Spring 2025
*Barrier Functions*

**Maegan Tucker**

April 3, 2025

Goals of Lecture 23

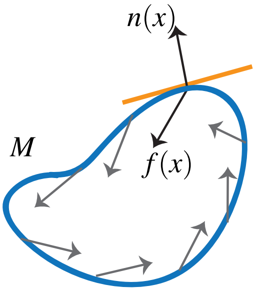
- Introduce Comparison Lemma
- Define Barrier Functions

Additional Reading

- A. Ames, S. Coogan, M. Egerstedt, G. Notomista, K. Sreenath, P. Tabuada, "Control Barrier Functions: Theory and Applications," IEEE Transactions on Automatic Control, 2019

# Barrier Functions

For $\dot{x} = f(x)$, recall from lecture 5 that we can check positive invariance of a set $\mathcal{C}$ by checking that $n(x)^T f(x) \leq 0$ for all $x$ on the boundary of $\mathcal{C}$ where $n(x)$ is an outward pointing normal vector to the set $\mathcal{C}$.



# Barrier Functions

If $\mathcal{C} = \{x \mid h(x) \geq 0\}$ for some continuously differentiable function $h$, then $n(x) = -\nabla h(x)$ whenever $\nabla h(x) \neq 0$, and then the previous condition becomes:

$$\nabla h(x)^T f(x) \geq 0 \text{ for all } x \text{ such that } h(x) = 0.$$

However, there are (at least) two potential problems with this approach:

- What if we have a function for which $\nabla h(x) = 0$ for some $x$ on the boundary of $\mathcal{C}$?
- Above condition is only at the boundary and is not good for creating controllers (everything is fine until suddenly it's not)

# Barrier Functions

Intuitive idea of barriers: make sure the system "slows down" as it approaches the boundary of $\mathcal{C}$.

- This lecture: barrier functions for autonomous systems
- Next lecture: control barrier functions for control-affine systems

# Barrier Function: A Definition

Definition: A function $h$ with $\mathcal{C} = \{x \mid h(x) \geq 0\}$ is a *barrier function* for $\dot{x} = f(x)$ if there exists a locally Lipschitz function $\alpha : \mathbb{R} \to \mathbb{R}$ satisfying $\alpha(0) = 0$ such that

$$\nabla h(x)^T f(x) \geq -\alpha(h(x)) \quad \text{for } \textit{all } x \in \mathbb{R}^n.$$

Using Lie derivative notation, recall $\nabla h(x)^T f(x) = L_f h(x) = \dot{h}(x)$.

- In general, we think of $\alpha$ as being an increasing function, but this is not needed for the theory on the next slide.
- Discussion of "local Lipschitz" requirement at end of lecture.

- When $\alpha$ is also increasing, it is sometimes called an *extended class $\mathcal{K}$ function*; recall our definition of class $\mathcal{K}$ functions from Lecture 12

# Invariance from Barrier Functions

Theorem: If $h$ is a barrier function, then $\mathcal{C} = \{x : h(x) \geq 0\}$ is positively invariant.

Proof relies on the following lemma:

Lemma (Comparison lemma): Consider the scalar system

$$\dot{z} = g(z), \quad z(0) = z_0$$

with locally Lipschitz $g$. Let $v(t)$ be some continuously differentiable function satisfying

$$\dot{v}(t) \geq g(v(t)) \text{ for all } t \geq 0, \text{ and}$$

$$v(0) \geq z_0.$$

Then $v(t) \geq z(t)$ for all $t$.

# Proof of Theorem

Proof of barrier theorem:

1. Let $x(t)$ be any system trajectory such that $x(0) \in \mathcal{C}$ and define $v(t) = h(x(t))$. Then $\dot{v}(t) = \nabla h(x(t))^T f(x(t)) \geq -\alpha(h(x(t)) = -\alpha(v(t))$, *i.e.*,
$$\dot{v}(t) \geq -\alpha(v(t)).$$
2. Note that $z(t) \equiv 0$ is a trajectory of $\dot{z} = -\alpha(z)$ since the initial condition $z(0) = 0$ is an equilibrium. Since $v(t) \geq z(0)$, by the Comparison Lemma, $v(t) \geq z(t) = 0$ for all $t \geq 0$, which means $x(t) \in \mathcal{C}$ for all $t \geq 0$.

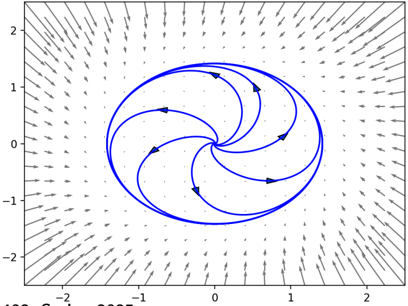
# Example

Example: Consider

$$\dot{x}_1 = (a - (x_1^2 + x_2^2))x_1 - x_2$$
$$\dot{x}_2 = (a - (x_1^2 + x_2^2))x_2 + x_1.$$

In polar coordinates,

$$\dot{r} = r(a - r^2), \quad \dot{\theta} = 1.$$



# Example (cont)

Let $C = \{x : h(x) \geq 0\}$ with $h(x) = a - (x_1^2 + x_2^2)$. Then

$$\dot{h}(x) =$$

$$\dot{x}_1 = (a - (x_1^2 + x_2^2))x_1 - x_2$$
$$\dot{x}_2 = (a - (x_1^2 + x_2^2))x_2 + x_1.$$

- In polar coordinates,

$$\dot{r} = r(a - r^2),$$
$$\dot{\theta} = 1.$$

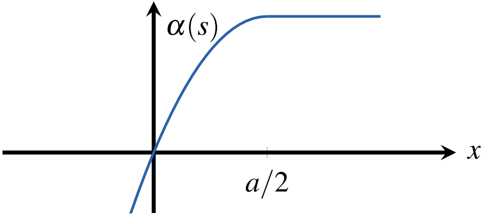
# Example (cont)

Let $C = \{x : h(x) \geq 0\}$ with $h(x) = a - (x_1^2 + x_2^2)$. Then

$$\dot{h}(x) = \nabla h(x)^T f(x) = 2(h(x) - a)h(x).$$

Take

$$\alpha(s) = \begin{cases} -2(s-a)s & \text{if } s \leq a/2 \\ a^2/2 & \text{if } s > a/2 \end{cases}$$




$$\dot{x}_1 = (a - (x_1^2 + x_2^2))x_1 - x_2$$
$$\dot{x}_2 = (a - (x_1^2 + x_2^2))x_2 + x_1.$$

- In polar coordinates,

$$\dot{r} = r(a - r^2),$$
$$\dot{\theta} = 1.$$

# Example 2

Example: Suppose $V(x)$ is a Lyapunov function for the system $\dot{x} = f(x)$. Take $h(x) = C - V(x)$ for some $C$. Then

$$\mathcal{C} = \{x \mid h(x) \geq 0\} = \{x \mid V(x) \leq C\}$$

We take $\alpha(s) = 0$ and establish positive invariance for $\mathcal{C}$, a sub-level set of $V$. This choice of $\alpha$ means that trajectories never move closer to the boundary of $\mathcal{C}$, as expected from Lyapunov theory.

# Example 3

Example: Consider

$$\dot{x}_1 = (-a + bx_2^2)x_1$$
$$\dot{x}_2 = (cx_1^2 - d)x_2.$$

We want to show that the union of the 1st and 3rd quadrants is invariant, *i.e.*, $C = \{(x_1, x_2) \mid h(x) \geq 0\}$ with $h(x) = x_1 x_2$. We have

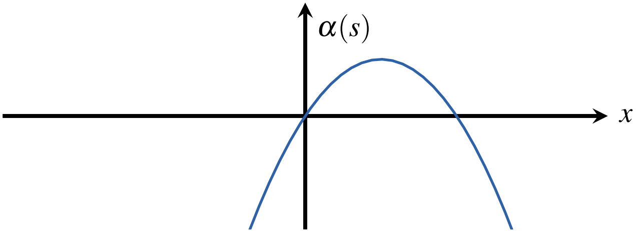
$$\begin{aligned}\dot{h}(x) = \nabla h(x)^T f(x) &= \dot{x}_1 x_2 + x_1 \dot{x}_2 \\ &= x_1 x_2(-a + bx_2^2) + x_1 x_2 (cx_1^2 - d).\end{aligned}$$

Note that, since $(\sqrt{b}x_2 - \sqrt{c}x_1)^2 \geq 0$, then $bx_2^2 + cx_1^2 \geq 2\sqrt{bc}x_1 x_2$ and therefore

$$\nabla h(x)^T f(x) \geq (-a - d + 2\sqrt{bc}h(x))h(x).$$

Take $\alpha(s) = -(-a - d + 2s\sqrt{bc})s$.

# Example 3 (cont.)

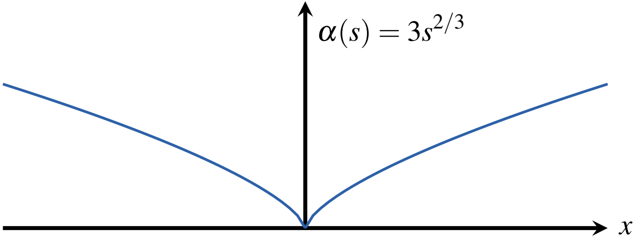



- Note that $\alpha(0) = 0$, as required. $\alpha$ is not increasing, but this is not an issue.

# Example 4

Local Lipschitzness of $\alpha$ is required for the Comparison Lemma to apply:

Example: Take $\dot{x} = -1$, $h(x) = x^3$ so $C = \{x : h(x) \geq 0\}$. Then $\dot{h}(x) = -h'(x) = -3x^2 = -3h(x)^{2/3}$. It is tempting to take $\alpha(s) = 3s^{2/3}$, a well-defined function satisfying $\alpha(0) = 0$, and it is even increasing for $s \geq 0$. But it is not Lipschitz, and the comparison lemma does not apply.



# Further discussion of Lipschitzness

- It is possible to weaken Lipschitz condition: The key is to ensure that, even if the comparison system $\dot{z} = -\alpha(z)$ with $z(0) = z_0$ has multiple solutions, all solutions remain nonnegative.
- An alternative assumption is to require that $\nabla h(x) \neq 0$ whenever $h(x) = 0$ so that $\nabla h(x)$ always provides a valid normal vector and our original technique (sometimes called Nagumo's theorem) applies. Then, the Comparison Lemma is not required.
- For this alternative, the proof of invariance does not require any other properties of $\alpha$ besides $\alpha(0) = 0$.
- Barriers are a hot topic, but beware that many papers fail to explicitly make either assumption.

- See R. Konda, A. Ames, S. Coogan, "Characterizing safety: minimal barrier functions from scalar comparison systems," IEEE Control Systems Letters, 2020, for more details