Lecture 22 – ME6402, Spring 2025
Important Classes of Convex Optimization Problems

Maegan Tucker

April 1, 2025

Georgia Tech

Goals of Lecture 22

- Introduce important classes of convex optimization problems

Additional Reading

- S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004.

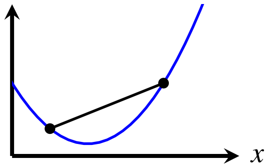
Recall from Lecture 21: Convex functions and sets

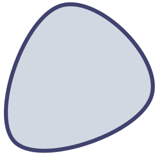
- A *convex function* $f : \mathbb{R}^n \to \mathbb{R}$ satisfies for all x, y and all $0 \leq \theta \leq 1$:

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y).$$

- A *convex set* $\mathcal{C}$ satisfies

  whenever $x_1, x_2 \in \mathcal{C}$, then $\theta x_1 + (1-\theta)x_2 \in \mathcal{C}$ for all $0 \leq \theta \leq 1$.





Example

The *probability simplex* is the set of vectors $x \in \mathbb{R}^n$ such that $x \geq 0$ and $\mathbf{1}^T x = 1$. It is a convex set:

Example

The *probability simplex* is the set of vectors $x \in \mathbb{R}^n$ such that $x \geq 0$ and $\mathbf{1}^T x = 1$. It is a convex set:

Let x_1 and x_2 be two elements of the probability simplex. For any $0 \leq \theta \leq 1$,

$$\theta x_1 + (1-\theta) x_2 \geq 0$$

and

$$\mathbf{1}^T (\theta x_1 + (1-\theta) x_2) = \theta \mathbf{1}^T x_1 + (1-\theta) \mathbf{1}^T x_2 = \theta + (1-\theta) = 1.$$

Example 2

The set of symmetric matrices in $\mathbb{R}^{n\times n}$ is a vector space (what is its dimension?). The subset of symmetric positive semidefinite matrices is a convex subset of this vector space. In fact, for any P.S.D. X_1 and X_2, and any $\theta_1 \geq 0$ and $\theta_2 \geq 0$, $\theta_1 X_1 + \theta_2 X_2$ is P.S.D.:

$$x^T(\theta_1 X_1 + \theta_2 X_2)x = \theta_1 \underbrace{x^T X_1 x}_{\geq 0} + \theta_2 \underbrace{x^T X_2 x}_{\geq 0} \geq 0$$

for all x.

Convex Optimization

Recall the optimization problem:

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0, \qquad i = 1, \ldots, m \end{aligned}$$

The above optimization problem is *convex* if f_0 and all f_i's are convex. In this case, the feasible set is a convex set.

Example

Example: Least Squares is a convex optimization problem:

$$\text{minimize} \quad \|Ax - b\|_2^2$$

We know $\|\cdot\|_2$ is convex because it is a norm (example in last lecture), $\|\cdot\|_2^2$ is also convex (convince yourself of this), and composition with affine transformation preserves convexity (example in last lecture).

- This is an *unconstrained* optimization problem since there are no constraints.
- Optimization problems rarely have closed form solutions, but the least squares problem does: $x = (A^T A)^{-1} A^T b$

Linear Optimization Programs (LP)

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & a_i^T x \leq b_i, \qquad i = 1, \ldots, m. \end{aligned}$$

- Linear programs are a class of optimization problems that can be solved very efficiently
- If feasible set is compact, then vertices of feasible region contain optimal points

Quadratic Optimization Programs (QP)

Quadratic costs give rise to quadratic optimization problems

Quadratic programs (QP): Quadratic cost with affine constraints

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2} x^T P x + q^T x + r \\ \text{subject to} \quad & a_i^T x \leq b_i, \qquad i = 1, \ldots, m \end{aligned}$$

where P is positive semidefinite, $P \succeq 0$.

Quadratically Constrained Quadratic Optimization Programs (QCQP)

Quadratically constrained quadratic programs (QCQP):

Quadratic cost with quadratic constraints.

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}x^T P_0 x + q_0^T x + r_0 \\ \text{subject to} \quad & \frac{1}{2}x^T P_i x + q_i^T x + r_i, \qquad i = 1, \ldots, m \end{aligned}$$

where all P_i's are positive semidefinite.

Example: Least squares is a QP because $\|Ax - b\|_2^2 = x^T A^T A x - 2b^T A x + b^T b$ and $A^T A \succeq 0$.

Example: All LPs are QPs, all QPs are QCQPs.

Second-Order Cone Programs (SOCP)

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \qquad i = 1, \ldots, m \end{array}$$

Example:

- If all c_i's are zero, then SOCP reduces to QCQP.
- If all A_i's are zero, then SOCP reduces to LP.

Linear matrix inequalities (LMIs) and semidefinite programs (SDPs)

A twist: Instead of scalar inequality ($\leq$) in constraints, what if we allowed for matrix inequality ($\preceq$)?

First form:

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & x_1 F_1 + x_2 F_2 + \ldots + x_n F_n + G \preceq 0 \end{aligned}$$

where $F_1, \ldots, F_n$ and G are all symmetric matrices.

- The inequality above is called a *linear matrix inequality (LMI)*.
- An optimization problem is a *semidefinite program (SDP)* if the constraints are LMIs are the cost is linear
- When $F_1, \ldots, F_n$ and G are actually scalars, we recover a standard affine constraint $f^T x + g \leq 0$

Linear matrix inequalities (LMIs) and semidefinite programs (SDPs)

We check that the constraint $x_1 F_1 + x_2 F_2 + \ldots + x_n F_n + G \preceq 0$ leads to a convex feasible set:

Linear matrix inequalities (LMIs) and semidefinite programs (SDPs)

We check that the constraint $x_1 F_1 + x_2 F_2 + \ldots + x_n F_n + G \preceq 0$ leads to a convex feasible set: Let $x_1, x_2, \ldots, x_n$ and $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n$ be two sets satisfying the semidefinite inequality. Then

$$
\begin{aligned}
&(\theta x_1 + (1-\theta)\hat{x}_1)F_1 + \ldots (\theta x_n + (1-\theta)\hat{x}_n)F_n + G \\
&= \theta(x_1 F_1 + \ldots + x_n F_n + G) + (1-\theta)(\hat{x}_1 F_1 + \ldots + \hat{x}_n F_n + G) \\
&\preceq 0
\end{aligned}
$$

Linear matrix inequalities (LMIs) and semidefinite programs (SDPs)

- Multiple LMIs can be combined into one LMI via block diagonalization:

$$x_1 F_1 + x_2 F_2 + \ldots + x_n F_n + G \preceq 0, \text{ and}$$
$$x_1 \hat{F}_1 + x_2 \hat{F}_2 + \ldots + x_n \hat{F}_n + \hat{G} \preceq 0$$

is same as

$$x_1 \begin{bmatrix} F_1 & 0 \\ 0 & \hat{F}_1 \end{bmatrix} + \ldots + x_n \begin{bmatrix} F_n & 0 \\ 0 & \hat{F}_n \end{bmatrix} + \begin{bmatrix} G & 0 \\ 0 & \hat{G} \end{bmatrix} \succeq 0$$

Linear matrix inequalities (LMIs) and semidefinite programs (SDPs)

Second form:

$$\begin{aligned} \text{minimize} \quad & \operatorname{trace}(CX) \\ \text{subject to} \quad & \operatorname{trace}(A_i X) = b_i, \qquad i = 1, \ldots, m \\ & X \succeq 0. \end{aligned}$$

- These two forms can be shown to be equivalent.
- Seemingly more general constraints can be reduced to LMI constraints of the form above.
- In particular, matrix variables that appear linearly in semidefinite constraints are allowed.

Again check that the constraints trace$(A_i X) = b_i$, $X \succeq 0$ leads to a feasible set:

Linear matrix inequalities (LMIs) and semidefinite programs (SDPs)

Again check that the constraints $\text{trace}(A_i X) = b_i$, $X \succeq 0$ leads to a feasible set: Let X_1 and X_2 both be feasible. Then

$$\theta X_1 + (1-\theta) X_2 \succeq 0$$

and

$$\begin{aligned} &\text{trace}(A_i(\theta X_1 + (1-\theta)X_2)) \\ &= \theta \text{trace}(A_i X_1) + (1-\theta)\text{trace}(A_i X_2) = b_i \end{aligned}$$

for all $0 \leq \theta \leq 1$.

LMI Example

Example: The Lyapunov inequality is given by $L(X) = A^T X + XA$, and we know A is Hurwitz if and only if there exists $X \succ 0$ such that $L(x) \prec 0$. $L(X)$ is linear in X. To see that, consider $X = aX_1 + bX_2$ and notice that

$$\begin{aligned} L(X) &= A^T(aX_1 + bX_2) + (aX_1 + bX_2)A \\ &= a(A^T X_1 + X_1 A) + b(A^T X_2 + X_2 A) \\ &= aL(X_1) + bL(X_2). \end{aligned}$$

Thus $L(X) \preceq -\varepsilon I$ for some $\varepsilon > 0$ is a LMI constraint in the variable X.

Example:

Consider $\dot{x} = A(t)x$ where $A(t)$ switches from among the set $\{A_1, \ldots, A_m\}$

- Even if all A_i are Hurwitz, stability is not guaranteed.
- How could we prove asymptotic stability of $x = 0$?

Example:

Consider $\dot{x} = A(t)x$ where $A(t)$ switches from among the set $\{A_1, \ldots, A_m\}$

- Even if all A_i are Hurwitz, stability is not guaranteed.
- How could we prove asymptotic stability of $x = 0$?

One approach: Find a common Lyapunov function $V(x) = x^T P x$ that works for all A_is. Pose as SDP:

$$\begin{aligned} \text{minimize}_P \quad & \text{trace}(P) \\ \text{subject to} \quad & PA_i + A_i^T P \preceq -\varepsilon I, \quad i = 1, \ldots, m \\ & P \succeq I. \end{aligned}$$

Solving convex optimization problems

Even though analytic solutions to convex optimization problems rarely exist, solvers have become so good and so fast that it is common to think of exact solutions to convex optimization problems as being readily available.

- CVX, CVXPY, CVXOPT, YALMIP are all basic purpose packages for solving convex optimization problems.
- Specialized functions such as MATLAB's `quadprog` for specific classes of problems (quadratic, in this case)

Solving convex optimization problems

Example. CVX provides easy coding of convex optimization problems in MATLAB:

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2 \\ \text{subject to} & Cx \leq d \end{array}$$

translated as

```
cvx_begin
    variable x(n)
    minimize( norm( A * x - b, 2 ) )
    subject to
        C * x <= d
cvx_end
```