Lecture 20 – ME6402, Spring 2025
Control Lyapunov Functions

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Goals of Lecture 20

- Define control lyapunov functions
- Present Sontag's universal formula for smooth stabilization
- Define small control property

Additional Reading

- E. Sontag, 1983
- Z. Arstein, 1978

Lyapunov for Control Systems

Lyapunov *analysis* of Lecture 9: For system

$$\dot{x} = f(x), \qquad f(0) = 0$$

find positive definite Lyapunov function $V(x)$ such that $\dot{V}(x)$ is negative definite to prove asym. stability of $x = 0$.

What about controlling for stability?

- An idea: For

  $$\dot{x} = f(x) + g(x)u$$

  and a candidate positive definite Lyapunov function $V(x)$, choose u such that $\dot{V}$ is negative definite

Control Lyapunov Function

Consider

$$\dot{x} = f(x) + g(x)u. \tag{1}$$

A positive definition function $V(x)$ is a (global) control Lyapunov function (CLF) for (1) if

$$\text{for all } x \neq 0, \text{ there exists } u \text{ s.t.} \quad \dot{V}(x) := \frac{\partial V}{\partial x}\left[f(x) + g(x)u\right] < 0.$$

Equivalently,

$$\frac{\partial V}{\partial x} g(x) = 0 \quad \text{and} \quad x \neq 0 \quad \Longrightarrow \quad \frac{\partial V}{\partial x} f(x) < 0.$$

- Be sure to convince yourself that these two statements are equivalent.

How to construct a controller from a CLF

If $u \in \mathbb{R}$, Sontag's formula:

$$u = \phi(x) = \begin{cases} -\left[\left(\frac{\partial V}{\partial x} f\right) + \sqrt{\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4}\,\right] / \left(\frac{\partial V}{\partial x} g\right) & \text{if } \left(\frac{\partial V}{\partial x} g\right) \neq 0 \\ 0 & \text{if } \left(\frac{\partial V}{\partial x} g\right) = 0 \end{cases}$$

- Choosing $u = \phi(x)$ asymptotically stabilizes the origin. Proof is straightforward (next slide)
- Formula seems complicated. Why? (Examples)

Proof of Sontag Formula

Compute $\dot{V}(x)$ for $x \neq 0$:

- If $\frac{\partial V}{\partial x} g(x) = 0$, then

$$\dot{V}(x) \frac{\partial V}{\partial x} f(x) < 0.$$

for any $x \neq 0$ by definition of CLF

- If $\frac{\partial V}{\partial x} g(x) \neq 0$, then

$$\dot{V}(x) = \frac{\partial V}{\partial x} \left[f(x) + g(x)\phi(x) \right] = -\sqrt{\left(\frac{\partial V}{\partial x} f \right)^2 + \left(\frac{\partial V}{\partial x} g \right)^4} < 0$$

Therefore, $x \neq 0$ implies $\dot{V}(x) < 0$, which shows asymptotic stability.

- Why the squareroot? Why the quartic power?

- Sontag's formula:

$$u = \phi(x) = \begin{cases} -\left[\left(\frac{\partial V}{\partial x} f \right) + \sqrt{\left(\frac{\partial V}{\partial x} f \right)^2 + \left(\frac{\partial V}{\partial x} g \right)^4} \right] \Big/ \left(\frac{\partial V}{\partial x} g \right) & \text{if } \left(\frac{\partial V}{\partial x} g \right) \neq 0 \\ 0 & \text{if } \left(\frac{\partial V}{\partial x} g \right) = 0 \end{cases}$$

Example 1

Consider

$$\dot{x} = -x^3 + u.$$

with CLF $V(x) = \frac{1}{2}x^2$. Let's consider a few controllers:

1. $u \equiv 0$
2. $u = \phi(x)$ from Sontag's formula
3. Modify Sontag's formula by changing 4th power to 2nd power
4. Modify Sontag's formula by removing squareroot

Example 1 (cont.)

$$\dot{x} = -x^3 + u,$$

$$u = \phi(x) \text{ from Sontag's formula}$$

$$=$$

2. $u = \phi(x)$ from Sontag's formula

- Sontag's formula:

$$u = \phi(x) = \begin{cases} -\left[\left(\frac{\partial V}{\partial x}f\right) + \sqrt{\left(\frac{\partial V}{\partial x}f\right)^2 + \left(\frac{\partial V}{\partial x}g\right)^4}\right] / \left(\frac{\partial V}{\partial x}g\right) & \text{if } \left(\frac{\partial V}{\partial x}g\right) \neq 0 \\ 0 & \text{if } \left(\frac{\partial V}{\partial x}g\right) = 0 \end{cases}$$

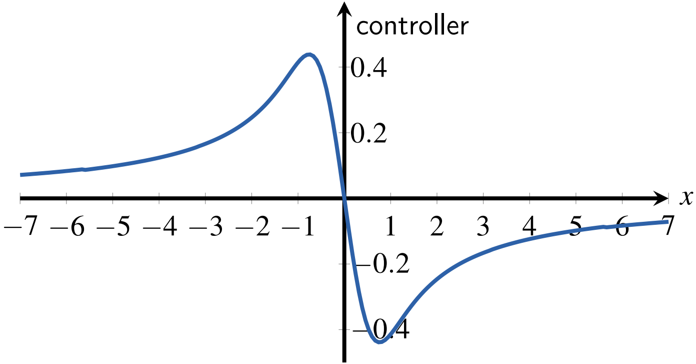
- Does Sontag's formula achieve exponential stability in this case?

Example 1 (cont.)

$$\dot{x} = -x^3 + u,$$

$$u = \phi(x) \text{ from Sontag's formula}$$

$$= x^3 - x\sqrt{x^4+1}$$



2. $u = \phi(x)$ from Sontag's formula

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3. Modify Sontag's formula by changing 4th power to 2nd power

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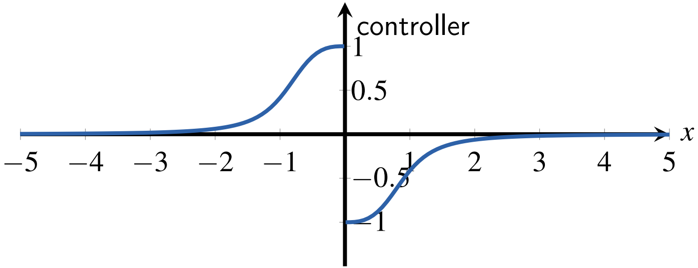
$$= -x^3 + \text{sign}(x)\sqrt{x^6 + 1}$$



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$$u = \begin{cases} -\left[\left(\frac{\partial V}{\partial x} f\right) + \left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4\right] / \left(\frac{\partial V}{\partial x} g\right) & \text{if } \left(\frac{\partial V}{\partial x} g\right) \neq 0 \\ 0 & \text{if } \left(\frac{\partial V}{\partial x} g\right) = 0 \end{cases}$$

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4. Modify Sontag's formula by removing squareroot

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$$= -x^7$$

- Extreme control action as x moves away from 0

4. Modify Sontag's formula by removing squareroot

What Else Can Go Wrong?

Example: $\dot{x} = x + x^2 u$ with CLF $V(x) = \frac{1}{2}x^2$.
Sontag's formula:

$$\phi(x) =$$

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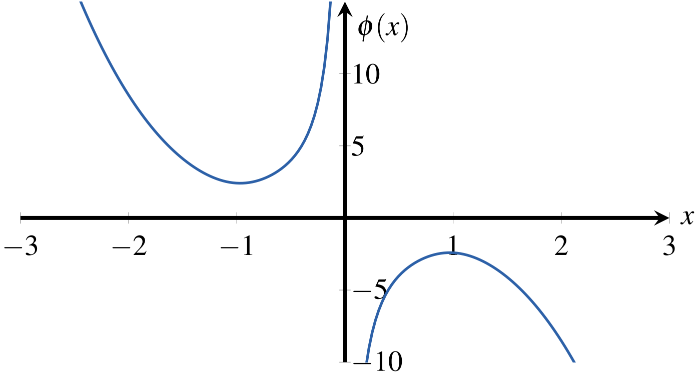
What Else Can Go Wrong?

Example: $\dot{x} = x + x^2 u$ with CLF $V(x) = \frac{1}{2}x^2$.

Sontag's formula:

$$\phi(x) = \frac{-1 - \sqrt{1 + x^8}}{x}$$



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Small Control Property

<u>Definition:</u> The system $\dot{x} = f(x) + g(x)u$ along with the CLF $V(x)$ satisfies the *small control property* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \neq 0$ and $|x| < \delta$, then there is u with $|u| < \varepsilon$ such that

$$\frac{\partial V}{\partial x}[f(x) + g(x)u] < 0.$$

Informally, small control effort is required to stay near the equilibrium $x = 0$.

- Sufficient condition for small control property is that there exists *some* Lipschitz continuous, stabilizing controller $u = k(x)$ such that $V(x)$ is a Lyapunov function for $\dot{x} = f(x) + g(x)k(x)$.

CLF Theorem

<u>Theorem.</u> If $f(x)$, $g(x)$, and $V(x)$ are continuously differentiable, then controller $\phi(x)$ from Sontag's formula is continuously differentiable for $x \neq 0$. If, further, $V(x)$ satisfies small control property, then $\phi(x)$ is also continuous at $x = 0$.

Sontag's Formula vs Feedback Linearization: Example

Consider

$$\dot{x} = x - x^3 + u.$$

- Feedback linearization:

$$u = k(x) = -x + x^3 - \alpha x, \quad \alpha > 0$$

  $V(x) = \frac{1}{2}x^2$ is a Lyapunov function.
- Sontag's formula using same $V(x)$:

$$u = \phi(x) =$$

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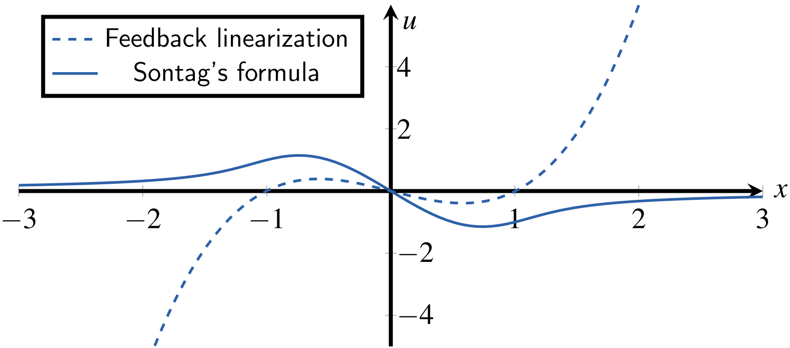
- Sontag's formula using same $V(x)$:

$$u = \phi(x) = \frac{x(-x+x^3) - x\sqrt{x^2(x-x^3)^2 + x^4}}{x}$$
$$= -x + x^3 - \sqrt{(1-x^2)^2 + 1}$$

Sontag's Formula vs Feedback Linearization: Example (cont.)



- Sontag's formula can keep useful nonlinearities (like $-x^3$), while feedback linearization cancels all nonlinearities
- No universal theorem that Sontag's formula is always "better"

Robustness of Sontag's formula

Fact: if $\phi(x)$ is from Sontag's formula, then $u = k\phi(x)$ stabilizes $x = 0$ for any $k \geq 1/2$, and same $V(x)$ is a Lyapunov function. That is, Sontag's formula has a gain margin of $[1/2, \infty)$.