Lecture 2 – ME6402, Spring 2025

Essentially Nonlinear Phenomena

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Goals of Lecture 2

- List several phenomena unique to systems that are not linear
- Phase portraits in the plane

Additional Reading

- Khalil, Chapter 2

Essentially Nonlinear Phenomena

1. Finite Escape Time

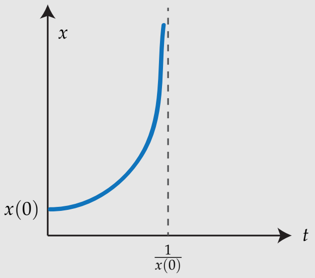
Example: $\dot{x} = x^2$

$$\frac{d}{dt}x^{-1} = -x^{-2}\dot{x} = -1$$

$$\Rightarrow \quad \frac{1}{x(t)} - \frac{1}{x(0)} = -t$$

$$\Rightarrow \quad x(t) = \frac{1}{\frac{1}{x(0)} - t}$$

For linear systems, $x(t) \to \infty$ cannot happen in finite time.



Essentially Nonlinear Phenomena

2. Multiple Isolated Equilibria
   Linear systems: either unique equilibrium or a continuum
   Pendulum: two isolated equilibria (one stable, one unstable)
   "Multi-stable" systems: two or more stable equilibria

Essentially Nonlinear Phenomena: Ex. of Multiple Isolated Equilibria

Example: bistable switch

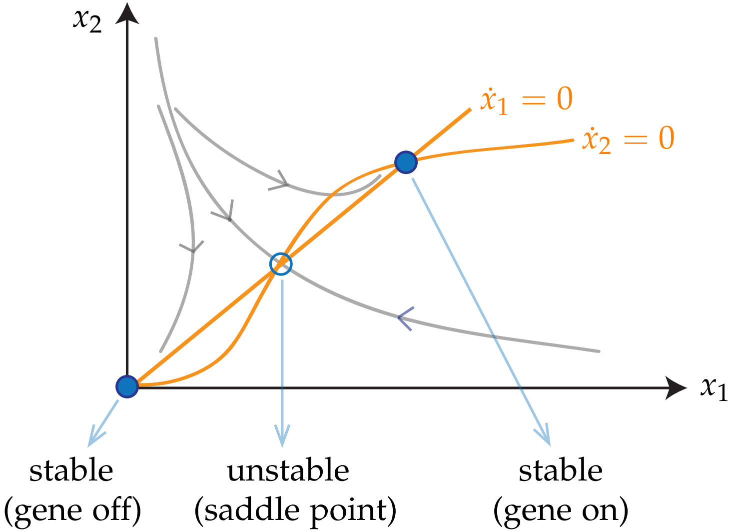
$$\dot{x}_1 = -ax_1 + x_2 \qquad x_1: \text{ concentration of protein}$$
$$\dot{x}_2 = \frac{x_1^2}{1+x_1^2} - bx_2 \qquad x_2: \text{ concentration of mRNA}$$

$a > 0$, $b > 0$ are constants. State space: $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$.

This model describes a positive feedback where the protein encoded by a gene stimulates more transcription via the term $\frac{x_1^2}{1+x_1^2}$.

Single equilibrium at the origin when $ab > 0.5$. If $ab < 0.5$, the line where $\dot{x}_1 = 0$ intersects the sigmoidal curve where $\dot{x}_2 = 0$ at two other points, giving rise to a total of three equilibria.

Essentially Nonlinear Phenomena: Ex. of Multiple Isolated Equilibria (cont.)

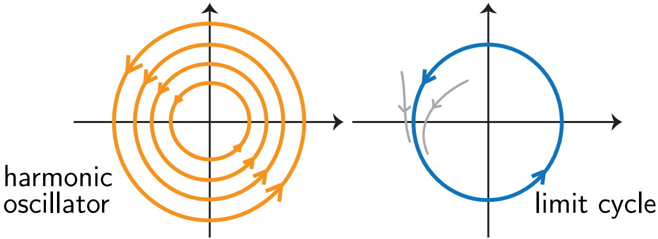


$$\dot{x}_1 = -ax_1 + x_2$$
$$\dot{x}_2 = \frac{x_1^2}{1+x_1^2} - bx_2$$

3. Limit cycles: Linear oscillators exhibit a continuum of periodic orbits; *e.g.*, every circle is a periodic orbit for $\dot{x} = Ax$ where

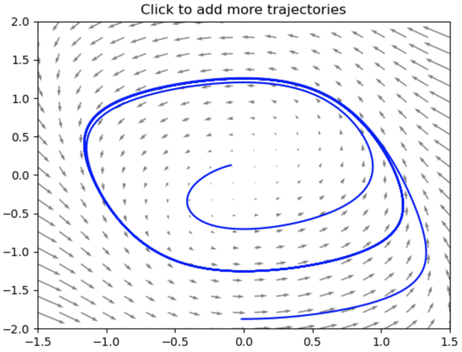
$$A = \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix} \qquad (\lambda_{1,2} = \mp j\beta).$$

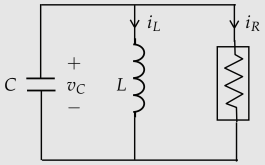
In contrast, a limit cycle is an isolated periodic orbit and can occur only in nonlinear systems.

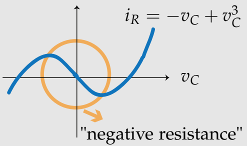


Example: van der Pol oscillator

$$C\dot{v}_C = -i_L + v_C - v_C^3$$
$$L\dot{i}_L = v_C$$



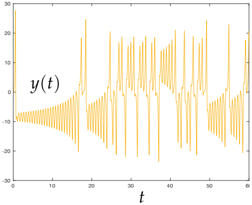


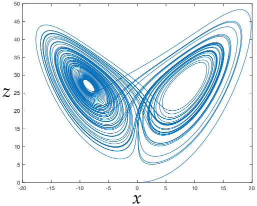


4. Chaos: Irregular oscillations, never exactly repeating.
   Example: Lorenz system (derived by Ed Lorenz in 1963 as a simplified model of convection rolls in the atmosphere):

$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= rx-y-xz\\ \dot{z} &= xy-bz.\end{aligned}$$

   Chaotic behavior with $\sigma = 10,\ b = 8/3,\ r = 28$:

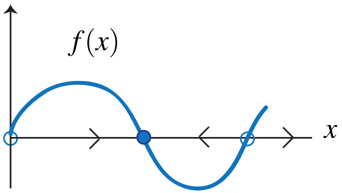




Essentially Nonlinear Phenomena: Chaos (cont.)

- For continuous-time, time-invariant systems, $n \geq 3$ state variables required for chaos.

  $\underline{n = 1:}$ $x(t)$ monotone in t, no oscillations:

  $\underline{n = 2:}$ Poincaré-Bendixson Theorem (to be studied in Lecture 4) guarantees regular behavior.

Essentially Nonlinear Phenomena: Chaos (cont.)

- Poincaré-Bendixson does not apply to time-varying systems and $n \geq 2$ is enough for chaos (for Van der Pol oscillator can exhibit chaos).
- For discrete-time systems, $n = 1$ is enough (we will see an example in Lecture 6).

Planar (Second Order) Dynamical Systems: Linear

Phase Portraits of Linear Systems: $\dot{x} = Ax$

- Distinct real eigenvalues

$$T^{-1}AT = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

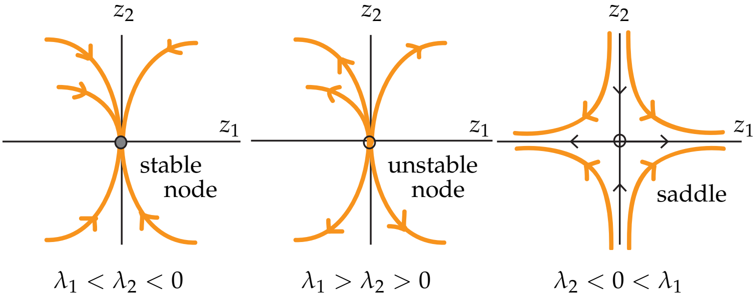
In $z = T^{-1}x$ coordinates:

$$\dot{z}_1 = \lambda_1 z_1, \quad \dot{z}_2 = \lambda_2 z_2.$$

- Chapter 2 in both Sastry and Khalil

Phase Portraits of Linear Systems, Real Eigenvalues (cont.)

The equilibrium is called a *node* when λ_1 and λ_2 have the same sign (*stable* node when negative and *unstable* when positive). It is called a *saddle point* when λ_1 and λ_2 have opposite signs.



Phase Portraits of Linear Systems, Complex Eigenvalues

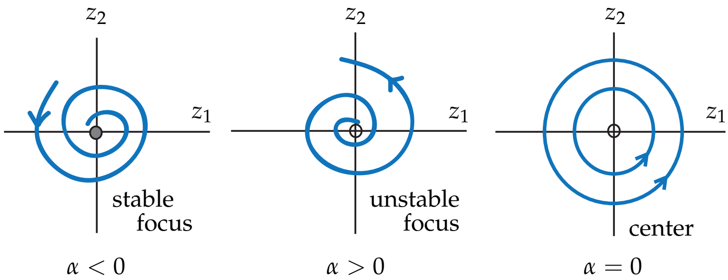
- Complex eigenvalues: $\lambda_{1,2} = \alpha \mp j\beta$

$$T^{-1}AT = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

$$\begin{aligned} \dot{z}_1 &= \alpha z_1 - \beta z_2 \\ \dot{z}_2 &= \alpha z_2 + \beta z_1 \end{aligned} \quad \rightarrow \quad \text{polar coordinates} \quad \rightarrow \quad \begin{aligned} \dot{r} &= \alpha r \\ \dot{\theta} &= \beta \end{aligned}$$

Phase Portraits of Linear Systems, Complex Eigenvalues (cont.)

Complex eigenvalues:



The phase portraits above assume $\beta > 0$ so that the direction of rotation is counter-clockwise: $\dot{\theta} = \beta > 0$.