Lecture 19 – ME6402, Spring 2025
Full-state Feedback Linearization (cont.)

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Goals of Lecture 19

- Feedback linearization for MIMO Systems
- Control of Drift-free systems

Additional Reading

- Khalil, Chapter 13
- Sastry, Chapter 9

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Feedback Linearization Continued

Recall "strict feedback systems" discussed in Lecture 14:

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\
\dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)x_3 \\
\dot{x}_3 &= f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)x_4 \\
&\vdots \\
\dot{x}_n &= f_n(x) + g_n(x)u.
\end{aligned}$$

Such systems are feedback linearizable when $g_i(x_1, \dots, x_i) \neq 0$ near the origin, $i = 1, 2, \cdots, n$, because the relative degree is n with the choice of output $y = h(x) = x_1$:

$$y^{(n)} = L_f^n h(x) + \underbrace{g_1(x_1)g_2(x_1, x_2)\cdots g_n(x)}_{L_g L_f^{n-1} h(x) \neq 0} u.$$

Example

Feedback linearizability is lost when $g_i(0) = 0$ for some i; however, backstepping may be applicable:

<u>Example 1 (from Lecture 14):</u>

$$\begin{aligned} \dot{x}_1 &= x_1^2 x_2 \\ \dot{x}_2 &= u. \end{aligned}$$

Treat x_2 as virtual control and let $\alpha_1(x_1) = -x_1$ which stabilizes the x_1-subsystem, as seen with Lyapunov function $V_1(x_1) = \frac{1}{2}x_1^2$. Then $z_2 := x_2 - \alpha_1(x_1)$ satisfies $\dot{z}_2 = u - \dot{\alpha}_1$, and

$$u = \dot{\alpha}_1 - \frac{\partial V_1}{\partial x_1} x_1^2 - k_2 z_2 = -x_1^2 x_2 - x_1^3 - k_2(x_2 + x_1)$$

achieves global asymptotic stability:

$$V = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2 \quad \Rightarrow \quad \dot{V} = -{x_1}^4 - k_2 z_2^2.$$

- Backstepping: for system

$$\begin{aligned} \dot{\eta} &= F(\eta) + G(\eta)x \\ \dot{x} &= u, \end{aligned}$$

use as input

$$u = \dot{\alpha} - \frac{\partial V}{\partial \eta} G(\eta) - kz, \ \ k > 0$$

with modified Lyapunov function $V_+(\eta, z) = V(\eta) + \frac{1}{2}z^2$.

Example (cont.)

In contrast the system is not feedback linearizable, because condition (C1) in the theorem for feedback linearizability (Lecture 18, p.4) fails. To see this note that

$$\begin{aligned} \dot{x}_1 &= x_1^2 x_2 \\ \dot{x}_2 &= u. \end{aligned}$$

Theorem: $\dot{x} = f(x) + g(x)u$ is feedback linearizable around x_0 if and only if the following two conditions hold:

C1) $[g(x_0) \;\; \mathrm{ad}_f\, g(x_0) \;\; \dots \;\; \mathrm{ad}_f^{n-1} g(x_0)]$ has rank n

C2) $\Delta(x) = \mathrm{span}\{g(x), \mathrm{ad}_f\, g(x), \dots, \mathrm{ad}_f^{n-2} g(x)\}$ is involutive in a neighborhood of x_0.

Example (cont.)

In contrast the system is not feedback linearizable, because condition (C1) in the theorem for feedback linearizability (Lecture 18, p.4) fails. To see this note that

$$f(x) = \begin{bmatrix} x_1^2 x_2 \\ 0 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathrm{ad}_f\, g(x) = [f, g](x) = \begin{bmatrix} -x_1^2 \\ 0 \end{bmatrix},$$

thus, with $n = 2$ and $x_0 = 0$,

$$[g(x_0) \quad \mathrm{ad}_f\, g(x_0) \quad \dots \quad \mathrm{ad}_f^{n-1} g(x_0)] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

which is rank deficient.

$$\begin{aligned} \dot{x}_1 &= x_1^2 x_2 \\ \dot{x}_2 &= u. \end{aligned}$$

Theorem: $\dot{x} = f(x) + g(x)u$ is feedback linearizable around x_0 if and only if the following two conditions hold:

C1) $[g(x_0) \quad \mathrm{ad}_f\, g(x_0) \quad \dots \quad \mathrm{ad}_f^{n-1} g(x_0)]$ has rank n

C2) $\Delta(x) = \mathrm{span}\{g(x), \mathrm{ad}_f\, g(x), \dots, \mathrm{ad}_f^{n-2} g(x)\}$ is involutive in a neighborhood of x_0.

Multi-Input Multi-Output Systems

Consider now a MIMO system with m inputs and m outputs:

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x) u_i \\ y_i &= h_i(x), \quad i = 1, \cdots, m. \end{aligned}$$

Let r_i denote the number of times we need to differentiate y_i to hit at least one input. Then,

$$\begin{bmatrix} y_1^{(r_1)} \\ \vdots \\ y_m^{(r_m)} \end{bmatrix} = \underbrace{\begin{bmatrix} L_f^{r_1} h_1(x) \\ \vdots \\ L_f^{r_m} h_m(x) \end{bmatrix}}_{=:\, B(x)} + \underbrace{\begin{bmatrix} L_{g_1} L_f^{r_1 - 1} h_1(x) & \cdots & L_{g_m} L_f^{r_1 - 1} h_1(x) \\ \vdots & & \vdots \\ L_{g_1} L_f^{r_m - 1} h_m(x) & \cdots & L_{g_m} L_f^{r_m - 1} h_m(x) \end{bmatrix}}_{=:\, A(x)} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}.$$

Multi-Input Multi-Output Systems (cont.)

If $A(x)$ is nonsingular, then the feedback law

$$u = A(x)^{-1}(-B(x) + v)$$

input/output linearizes the system, creating m decoupled chains of integrators:

$$y_i^{(r_i)} = v_i, \quad i = 1, \ldots, m.$$

We say that the system has *vector relative degree* $\{r_1, \cdots, r_m\}$ if the matrix $A(x)$ defined above is nonsingular.

$$\begin{bmatrix} y_1^{(r_1)} \\ \vdots \\ y_m^{(r_m)} \end{bmatrix} = \underbrace{\begin{bmatrix} L_f^{r_1} h_1(x) \\ \vdots \\ L_f^{r_m} h_m(x) \end{bmatrix}}_{=:B(x)} + \underbrace{\begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & \cdots & L_{g_m} L_f^{r_1-1} h_1(x) \\ \vdots & & \vdots \\ L_{g_1} L_f^{r_m-1} h_m(x) & \cdots & L_{g_m} L_f^{r_m-1} h_m(x) \end{bmatrix}}_{=:A(x)} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}.$$

Example 2: The kinematic model of a unicycle, depicted on the right, is

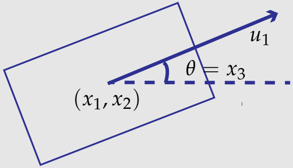
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2,$$

where u_1 is the speed and u_2 is the angular velocity.

Let $y_1 = x_1$ and $y_2 = x_2$, and note that

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos x_3 & 0 \\ \sin x_3 & 0 \end{bmatrix}}_{=: A(x)} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Since $A(x)$ is singular, the system does not have a well-defined vector relative degree. □



Multi-Input Multi-Output Systems

- The notion of zero dynamics and the normal form can be extended to MIMO systems. If the system has vector relative degree $\{r_1, \cdots, r_m\}$, then $r := r_1 + \cdots + r_m \leq n$ and
$$\zeta := [h_1(x)\ L_f h_1(x) \cdots L_f^{r_1-1} h_1(x)\ \cdots\ h_m(x)\ L_f h_m(x) \cdots L_f^{r_m-1} h_m(x)]^T$$
defines a partial set of coordinates.
- As in normal form discussed in Lecture 17, one can find $n-r$ additional functions $z_1(x), \cdots, z_{n-r}(x)$ so that $x \mapsto (z, \zeta)$ is a complete coordinate transformation.
- Full-state feedback linearization amounts to finding m output functions $h_1, \cdots, h_m$ such that the system has vector relative degree $\{r_1, \cdots, r_m\}$ with $r_1 + \cdots + r_m = n$. Necessary and sufficient conditions for the existence of such functions, analogous to those in Lecture 18 for SISO systems, are available.

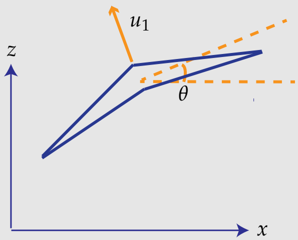
- See, see, *e.g.*, Sastry, Section 9.3 for MIMO zero dynamics
- See, *e.g.*, Sastry, Proposition 9.16 for full-state feedback linearization conditions

Example

Example 3: Consider the following model of a *planar vertical take-off and landing* (PVTOL) aircraft

$$\begin{aligned} \ddot{x} &= -\sin(\theta)u_1 + \mu\cos(\theta)u_2 \\ \ddot{z} &= \cos(\theta)u_1 + \mu\sin(\theta)u_2 - 1 \\ \ddot{\theta} &= u_2, \end{aligned}$$

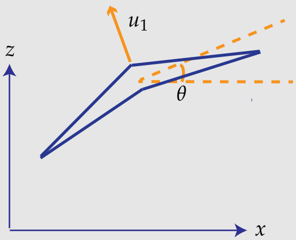
where μ is a constant that accounts for the coupling between the rolling moment and translational acceleration, and -1 in the second equation is the gravitational acceleration, normalized to unity by appropriately scaling the variables.



- Sastry, Section 10.4.2

Example (cont.)

If we take x and z as the two outputs we get



$$
\begin{aligned}
\ddot{x} &= -\sin(\theta)u_1 + \mu\cos(\theta)u_2 \\
\ddot{z} &= \cos(\theta)u_1 + \mu\sin(\theta)u_2 - 1 \\
\ddot{\theta} &= u_2,
\end{aligned}
$$

Example (cont.)

If we take x and z as the two outputs we get

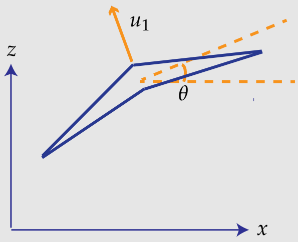
$$\begin{bmatrix} \ddot{x} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \underbrace{\begin{bmatrix} -\sin\theta & \mu\cos\theta \\ \cos\theta & \mu\sin\theta \end{bmatrix}}_{A(\theta)} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

where $A(\theta)$ is invertible when $\mu \neq 0$:

$$A^{-1}(\theta) = \begin{bmatrix} -\sin\theta & \cos\theta \\ \frac{1}{\mu}\cos\theta & \frac{1}{\mu}\sin\theta \end{bmatrix}.$$

Thus the systems has vector relative degree $\{2,2\}$ when $\mu \neq 0$, and the input/output linearizing controller is

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -\sin\theta & \cos\theta \\ \frac{1}{\mu}\cos\theta & \frac{1}{\mu}\sin\theta \end{bmatrix} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right).$$



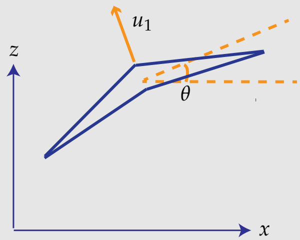
$$\begin{aligned} \ddot{x} &= -\sin(\theta)u_1 + \mu\cos(\theta)u_2 \\ \ddot{z} &= \cos(\theta)u_1 + \mu\sin(\theta)u_2 - 1 \\ \ddot{\theta} &= u_2, \end{aligned}$$

Example (cont.)

The zero dynamics is obtained by substituting $u_2^* = \frac{1}{\mu}\sin\theta$, needed to maintain z at a constant value and $\dot{z}$ at zero, in the dynamical equation for θ:

$$\ddot{\theta} = \frac{1}{\mu}\sin\theta.$$

The system is nonminimum phase for $\mu > 0$, since $\theta = 0$ is unstable.



$$\begin{aligned}
\ddot{x} &= -\sin(\theta)u_1 + \mu\cos(\theta)u_2 \\
\ddot{z} &= \cos(\theta)u_1 + \mu\sin(\theta)u_2 - 1 \\
\ddot{\theta} &= u_2,
\end{aligned}$$

Drift-Free Systems

Suppose $f(x) = 0$ for all x in (2). Such system are called *drift-free* and encompass linear systems of the form

$$\dot{x} = Bu, \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m.$$

Assuming the columns of the $n \times m$ matrix B are linearly independent, we can find $n-m$ row vectors T_i, $i = 1, \cdots, n-m$, such that

$$T_i B = 0.$$

This means that $\phi_i(x) := T_i x$ satisfies

$$\frac{d}{dt}\phi_i(x(t)) = 0 \quad \Rightarrow \quad \phi_i(x(t)) = \phi_i(x(0))$$

regardless of the control inputs. Since there are $n-m$ such constraints, controllability is not possible in drift-free linear systems with fewer control inputs than the state dimension ($m < n$).

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i \qquad (2)$$

$$y_i = h_i(x), \quad i = 1, \cdots, m.$$

Drift-Free Systems (cont.)

The Frobenius Theorem (Lecture 18) implies that constraints satisfying $\frac{d}{dt}\phi_i(x(t)) = 0$, called *holonomic constraints*, also exist for nonlinear drift-free systems

$$\dot{x} = \sum_{i=1}^{m} g_i(x) u_i$$

when the distribution $\Delta = \text{span}\{g_1, \cdots, g_m\}$ is nonsingular and involutive.

$$\frac{d}{dt}\phi_i(x(t)) = 0$$

$$\Rightarrow \quad \phi_i(x(t)) = \phi_i(x(0))$$

Drift-Free Systems (cont.)

When $\Delta = \text{span}\{g_1, \cdots, g_m\}$ is non-involutive, however, controllability may be possible with $m < n$; this is another essentially nonlinear phenomenon.

Indeed, *Chow's Theorem* states that (3) is controllable if the *involutive closure* of $\Delta = \text{span}\{g_1, \cdots, g_m\}$ has dimension n. This condition means that the Lie brackets of $g_1, \cdots, g_m$ span new dimensions that are not already spanned by these basis vector fields.

- Drift-free systems satisfying Chow's Theorem are called *nonholonomic*.

- Drift-free nonlinear system:

$$\dot{x} = \sum_{i=1}^{m} g_i(x) u_i \tag{3}$$

- *Involutive closure* is the smallest involutive distribution that containts Δ

Example

Example 4: Recall the unicycle model discussed in Example 2, where

$$g_1(x) = \begin{bmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad [g_1, g_2](x) = \begin{bmatrix} -\sin x_3 \\ \cos x_3 \\ 0 \end{bmatrix}.$$

$\Delta = \mathrm{span}\{g_1, g_2\}$ is non-involutive, as $[g_1, g_2]$ generates a new direction. Taken together, g_1, g_2, and $[g_1, g_2]$ span the entire three-dimensional space at each point x; therefore, the system is controllable by Chow's Theorem. This conclusion sheds light on how parallel parking is possible despite lack of sideways actuation.

□

Interpreting Lie Brackets in Driftless Systems

To present an interpretation of the Lie bracket $[g_1, g_2]$, we let $\Phi_t^{g_i}(x_0)$ denote the solution of the system $\dot{x} = g_i(x)$ at time t from initial condition x_0. Then it can be shown that

$$\Phi_t^{-g_2}(\Phi_t^{-g_1}(\Phi_t^{g_2}(\Phi_t^{g_1}(x_0)))) = t^2[g_1, g_2](x_0) + \mathcal{O}(t^3),$$

which suggests that motion in the direction of the Lie bracket $[g_1, g_2]$ can be generated by alternating actuation of the two inputs u_1 and u_2 with positive and negative signs, as one does in parallel parking.