Lecture 17 – ME6402, Spring 2025
Feedback Linearization (continued)

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Goals of Lecture 17

- Normal form for input-output feedback linearizable systems

Additional Reading

- Khalil, Chapter 13
- Sastry, Chapter 9

Recall: Relative Degree

Consider the single-input single-output (SISO) nonlinear system:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x). \end{aligned} \tag{1}$$

Relative degree The system (1) has *relative degree* r if, in a neighborhood of the equilibrium,

$$\begin{aligned} &L_g L_f^{i-1} h(x) = 0 \quad i = 1, 2, \ldots, r-1 \\ &L_g L_f^{r-1} h(x) \neq 0. \end{aligned}$$

Recall:

$$\dot{y} = \underbrace{\frac{\partial h}{\partial x} f(x)}_{=: L_f h(x)} + \underbrace{\frac{\partial h}{\partial x} g(x)}_{=: L_g h(x)} u, \qquad \ddot{y} = \underbrace{L_f L_f h(x)}_{=: L_f^2 h(x)} + L_g L_f h(x) u.$$

- $L_f h$ is called the *Lie derivative* of h along the vector field f

Recall: Input-Output Linearization

If a system has a well-defined relative degree then it is input-output linearizable:

$$y^{(r)} = L_f^r h(x) + \underbrace{L_g L_f^{r-1} h(x)}_{\neq 0} u$$

Apply preliminary feedback:

$$\boxed{u = \frac{1}{L_g L_f^{r-1} h(x)} \left(-L_f^r h(x) + v \right)}$$

where v is a new input to be designed.

Choose any linear stabilizing controller that stabilizes integrator chain $y^{(r)} = v$.

Recall: Zero Dynamics

I/O Linearization+control renders the $(n-r)$-dimensional manifold:

$$h(x) = L_f h(x) = \cdots = L_f^{r-1} h(x) = 0$$

invariant and attractive.

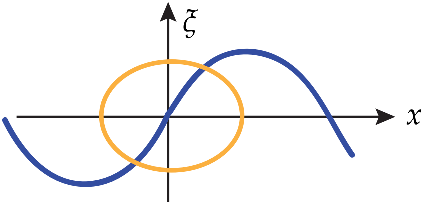
- The dynamics restricted to this manifold are called zero dynamics and determine whether or not $x = 0$ is stable.
- If the origin of the zero dynamics is asymptotically stable, the system is called minimum phase. If unstable, it is called nonminimum phase.

Nonlinear Changes of Variables

$T : \mathbb{R}^n \to \mathbb{R}^n$ is called a *diffeomorphism* if its inverse T^{-1} exists, and both T and T^{-1} are continuously differentiable (C^1).

Examples:

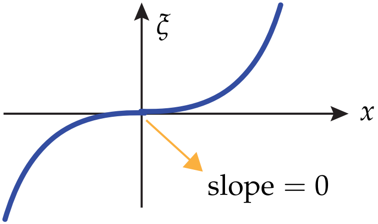
1. $\xi = Tx$ is a diffeomorphism if T is a nonsingular matrix
2. $\xi = \sin x$ is a local diffeomorphism around $x = 0$, but not global



Nonlinear Changes of Variables

Examples (cont):

3. $\xi = x^3$ is not a diffeomorphism because $T^{-1}(\cdot)$ is not C^1 at $\xi = 0$



Nonlinear Changes of Variables

How to check if $\xi = T(x)$ is a local diffeomorphism?

Implicit Function Theorem

Suppose $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is C^1 and there exists $x_0 \in \mathbb{R}^n$, $\xi_0 \in \mathbb{R}^m$ such that

$$f(x_0, \xi_0) = 0.$$

If $\dfrac{\partial f}{\partial x}(x_0, \xi_0)$ is nonsingular, then in a neighborhood of (x_0, ξ_0),

$$f(x, \xi) = 0$$

has a unique solution $x = g(\xi)$ where g is C^1 at $\xi = \xi_0$.

Corollary: Let $f(x, \xi) = T(x) - \xi$. If $\dfrac{\partial T}{\partial x}$ is nonsingular at x_0, then $T(\cdot)$ is a local diffeomorphism around x_0.

A "Normal Form" that Explicitly Displays the Zero Dynamics

Theorem: If $\dot{x} = f(x) + g(x)u$, $y = h(x)$ has a well-defined relative degree $r \leq n$, then there exist a diffeomorphism $T : x \to \begin{bmatrix} z \\ \zeta \end{bmatrix}$, $z \in \mathbb{R}^{n-r}$, $\zeta \in \mathbb{R}^r$, that transforms the system to the form:

$$
\begin{aligned}
\dot{z} &= f_0(z, \zeta) \\
\dot{\zeta}_1 &= \zeta_2 \\
&\vdots \\
\dot{\zeta}_r &= b(z, \zeta) + a(z, \zeta)u, \quad y = \zeta_1.
\end{aligned}
$$

In particular, $\dot{z} = f_0(z, 0)$ represents the zero dynamics. $\square$

A "Normal Form" that Explicitly Displays the Zero Dynamics

To obtain this normal form, let $\zeta = [h(x) \quad L_f h(x) \quad \dots \quad L_f^{r-1} h(x)]^T$, and find $n-r$ independent variables z such that $\dot{z}$ does not contain u.

Note that the terms $b(z,\zeta)$ and $a(z,\zeta)$ correspond to $L_f^r(x)$ and $L_g L_f^{r-1} h(x)$ in the original coordinates.

$$
\begin{aligned}
\dot{z} &= f_0(z,\zeta) \\
\dot{\zeta}_1 &= \zeta_2 \\
&\vdots \\
\dot{\zeta}_r &= b(z,\zeta) + a(z,\zeta)u, \quad y = \zeta_1.
\end{aligned}
$$

Example

Example:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= \alpha x_3 + u\\ \dot{x}_3 &= \beta x_3 - u\\ y &= x_1.\end{aligned}$$

Let $\zeta_1 = x_1$, $\zeta_2 = x_2$, and note that $z = x_2 + x_3$ is independent of ζ_1, ζ_2, and $\dot{z}$ does not contain u. Thus, the normal form is:

$$\begin{aligned}\dot{z} &= (\alpha+\beta)x_3 = (\alpha+\beta)z - (\alpha+\beta)\zeta_2\\ \dot{\zeta}_1 &= \zeta_2\\ \dot{\zeta}_2 &= \alpha x_3 + u = \alpha z - \alpha\zeta_2 + u.\end{aligned}$$

I/O Linearizing Controller in the new coordinates (9):

$$u = \frac{1}{a(z,\zeta)}\Big(-b(z,\zeta)+v\Big) \tag{2}$$

$$v = -k_1\zeta_1 \cdots - k_r\zeta_r \tag{3}$$

where $k_1, \cdots, k_r$ are such that all roots of $s^r + k_r s^{r-1} + \cdots + k_2 s + k_1$ have negative real parts.

I/O Linearizing Controller (cont.)

Theorem: If $z = 0$ is locally exponentially stable for the zero dynamics $\dot{z} = f_0(z, 0)$, then (4)–(5) locally exponentially stabilizes $x = 0$.

Proof: Closed-loop system:

$$\dot{z} = f_0(z, \zeta)$$
$$\dot{\zeta} = A\zeta$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & \\ 0 & 0 & 1 & \dots & \\ & & & \ddots & \\ & & & & 1 \\ -k_1 & -k_2 & -k_3 & \dots & -k_r \end{bmatrix}$$

is Hurwitz.

$$u = \frac{1}{a(z, \zeta)}\Big(- b(z, \zeta) + v\Big) \quad (4)$$

$$v = -k_1\zeta_1 \cdots - k_r\zeta_r \quad (5)$$

Proof (cont.): The Jacobian linearization at $(z, \zeta) = 0$ is:

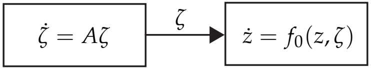
$$J = \begin{bmatrix} \frac{\partial f_0}{\partial z}(0,0) & \frac{\partial f_0}{\partial \zeta}(0,0) \\ 0 & A \end{bmatrix}$$

where $\frac{\partial f_0}{\partial z}(0,0)$ is Hurwitz since $\dot{z} = f_0(z,0)$ is exponentially stable by the proposition in Lecture 12. Since A is also Hurwitz, all eigenvalues of J have negative real parts $\Rightarrow$ exponential stability.

Global asymptotic stability can be guaranteed with additional assumptions on the zero dynamics, such as ISS of

$$\dot{z} = f_0(z, \zeta)$$

with respect to the input ξ:



$$u = \frac{1}{a(z,\zeta)}\Big(-b(z,\zeta) + v\Big)$$

$$v = -k_1\zeta_1 \cdots - k_r\zeta_r$$

Example

Example: $\dot{z} = -z + z^2\zeta, \quad \dot{\zeta} = -k\zeta$

$(z, \zeta) = 0$ is locally exponentially stable, but not globally: solutions escape in finite time for large $z(0)$.

- Note: the z subsystem is not ISS

I/O Linearizing Controller for Tracking

For the output $y(t)$ to track a reference signal $y_d(t)$, replace (5) with:

$$v = -k_1(\zeta_1 - y_d(t)) - k_2(\zeta_2 - \dot{y}_d(t)) \cdots - k_r(\zeta_r - y_d^{(r-1)}(t)) + y_d^{(r)}(t)$$

Let $e_1 \triangleq \zeta_1 - y_d(t)$, $e_2 \triangleq \zeta_2 - \dot{y}_d(t)$, $\ldots$, $e_r \triangleq \zeta_r - y_d^{(r-1)}(t)$. Then:

$$\left.\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= e_3 \\ &\vdots \\ \dot{e}_r &= v - y_d^{(r)}(t) = -k_1 e_1 - \cdots - k_r e_r \end{aligned}\right\} \quad \dot{e} = Ae.$$

Thus $e(t) \to 0$, that is $y(t) - y_d(t) \to 0$.

If $y_d(t)$ and its derivatives are bounded, then $\zeta(t)$ is bounded. If the zero dynamics $\dot{z} = f_0(z, \zeta)$ is ISS with respect to ζ, then $z(t)$ is also bounded. Thus, all internal signals are bounded.

- $y(t)$ is assumed to be r times differentiable