Lecture 15 – ME6402, Spring 2025
Input-to-State Stability

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February 27, 2025

Goals of Lecture 15

- Define input-to-state stability (ISS)
- Provide Lyapunov characterization of ISS

Additional Reading

- Khalil 4.9

Input-to-State Stability

$$\dot{x} = f(x,u) \quad u\text{: exogenous input}$$

- For linear systems, asymp. stability of the zero-input model $\dot{x} = Ax$ implies a bounded-input bounded-state property for $\dot{x} = Ax + Bu$

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$\implies |x(t)| \leq \|e^{At}\||x_0| + \int_0^t \|e^{A(t-\tau)}\|\|B\||u(\tau)|d\tau$$

$$\leq \kappa e^{-\alpha t}|x_0| + \|B\| \sup_{0\leq\tau\leq t} |u(\tau)| \int_0^t \kappa e^{-\alpha(t-\tau)}d\tau$$

$$\leq \underbrace{\kappa e^{-\alpha t}|x_0|}_{\text{effect of initial condition}} + \underbrace{\frac{\kappa}{\alpha}\|B\| \sup_{0\leq\tau\leq t} |u(\tau)|}_{\text{effect of input}}.$$

Input-to-State Stability

$$\dot{x} = f(x,u) \quad u\text{: exogenous input}$$

- For linear systems, asymp. stability of the zero-input model $\dot{x} = Ax$ implies a bounded-input bounded-state property for $\dot{x} = Ax + Bu$
- For nonlinear systems $\dot{x} = f(x,u)$, asymp. stability of the origin for the zero-input model $\dot{x} = f(x,0)$ does <u>not</u> guarantee boundedness of states under bounded inputs.

Example

Example 1: $\dot{x} = -x + xu$

$u(t) \equiv$ constant $> 1 \implies$ exponential growth of $x(t)$.

ISS Definition

A precise formulation of the bounded-input bounded-state property for nonlinear systems:

Definition: The system $\dot{x} = f(x,u)$, $f(0,0) = 0$ is said to be input-to-state stable (ISS) if:

$$|x(t)| \leq \beta(|x(0)|,t) + \gamma\left(\sup_{0\leq\tau\leq t} |u(\tau)|\right)$$

for some class-$\mathcal{KL}$ function β and class-$\mathcal{K}$ function γ, called an ISS gain function.

Example

Example: For the linear system, recall:

$$|x(t)| \leq \kappa e^{-\alpha t}|x_0| + \frac{\kappa}{\alpha}\|B\| \sup_{0\leq\tau\leq t} |u(\tau)|$$

so we can take

$$\beta(s,t) = \kappa e^{-\alpha t} s$$
$$\gamma(s) = \frac{\kappa}{\alpha}\|B\| s$$

- ISS condition:

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma\left(\sup_{0\leq\tau\leq t} |u(\tau)|\right)$$

Implications of ISS

1. $\dot{x} = f(x,u)$ ISS $\Longrightarrow$ $\dot{x} = f(x,0)$ globally asymptotically stable

   Proof:

   Substitute $u(t) \equiv 0$ in the definition above:

   $|x(t)| \leq \beta(|x(0)|,t)$.

Implications of ISS (cont.)

2. $u(t) \to 0$ as $t \to \infty \;\Rightarrow\; x(t) \to 0$ as $t \to \infty$.

   Proof: Need to show that for any $\varepsilon > 0$, there exists T such that
$$|x(t)| \le \varepsilon \quad \forall t \ge T.$$
   Since $u(t) \to 0$, we can find T_1 such that $\gamma(|u(t)|) \le \varepsilon/2$ for all $t \ge T_1$. Choose $t_0 = T_1$ and apply ISS definition:
$$|x(t)| \le \beta(|x(T_1)|, t - T_1) + \varepsilon/2 \quad \forall t \ge T_1.$$
   Choose T_2 such that
$$\beta(|x(T_1)|, T_2) \le \varepsilon/2.$$
   Then, $|x(t)| \le \varepsilon$ for all $t \ge T_1 + T_2 \triangleq T$.

A Lyapunov Characterization of ISS

The system $\dot{x} = f(x,u)$ is ISS if there exist class-$\mathcal{K}_\infty$ functions $\alpha_i,\ i = 1,2,3,4$, and a C^1 function V such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$

$$\frac{\partial V}{\partial x} f(x,u) \leq -\alpha_3(|x|) + \alpha_4(|u|).$$

V is called an "ISS Lyapunov function."

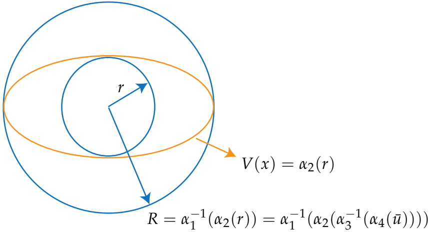
A Lyapunov Characterization of ISS (cont.)

Sketch of the proof:

Let $\bar{u} \triangleq \sup_{\tau \geq 0} |u(\tau)|$. Then:

$$|x| \geq r \triangleq \alpha_3^{-1}(\alpha_4(\bar{u})) \quad \Rightarrow \quad \frac{\partial V}{\partial x} f(x, u(t)) \leq 0 \quad \forall t \geq 0.$$

This implies that the level set $\{x : V(x) \leq \alpha_2(r)\}$ is invariant and attractive. Thus, all trajectories converge to this level set which is enclosed in the outer ball $|x| \leq R \triangleq \alpha_1^{-1}(\alpha_2(r))$.



Example 2

Example 2: $\dot{x} = -x^r + x^s u$, r: odd integer, is ISS if $r > s$. Take:

$$V(x) = \frac{1}{2}x^2$$
$$\dot{V}(x) = -x^{r+1} + x^{s+1}u.$$

Young's inequality:

$$yz \leq \frac{\lambda^p}{p}|y|^p + \frac{1}{q\lambda^q}|z|^q$$

for any $\lambda > 0$, and $p > 1, q > 1$ satisfying $(p-1)(q-1) = 1$.

Example 2 (cont.)

Apply Young's inequality to:

$$x^{s+1}u \le \frac{\lambda^p}{p}|x|^{(s+1)p} + \frac{1}{q\lambda^q}|u|^q$$

and choose

$$p = \frac{r+1}{s+1} \quad q = 1 + \frac{1}{p-1} \text{ and } \lambda \text{ such that } \frac{\lambda^p}{p} = \frac{1}{2}$$

$$\Rightarrow \; x^{s+1}u \le \frac{1}{2}|x|^{r+1} + \frac{1}{q\lambda^q}|u|^q$$

$$\Rightarrow \; \dot{V}(x) \le -|x|^{r+1} + \frac{1}{2}|x|^{r+1} + \frac{1}{q\lambda^q}|u|^q$$

$$\le \underbrace{-\frac{1}{2}|x|^{r+1}}_{-\alpha_3(|x|)} + \underbrace{\frac{1}{q\lambda^q}|u|^q}_{-\alpha_4(|u|)}.$$

- $\dot{x} = -x^r + x^s u$, r: odd integer, $r > s$.
- Young's inequality:

$$yz \le \frac{\lambda^p}{p}|y|^p + \frac{1}{q\lambda^q}|z|^q$$

for any $\lambda > 0$, and $p > 1, q > 1$ satisfying $(p-1)(q-1) = 1$.

Example 2 (cont.)

Note:

- $\dot{x} = -x + xu$ ($r = s = 1$) is not ISS as shown in Example 1.
- $\dot{x} = -x + x^2 u$ ($r = 1, s = 2$) is not ISS: it exhibits finite time escape for $u(t) \equiv$ constant $\neq 0$, even with an exponentially decaying $u(t)$.
- $\dot{x} = -x^3 + u$ ($r = 3, s = 0$) is ISS.

Example 3

$$\dot{x}_1 = -x_1 + x_2^2$$
$$\dot{x}_2 = -x_2 + u.$$

Let $V(x) = \frac{1}{2}x_1^2 + \frac{a}{4}x_2^4$, $a > 0$ to be determined.[1]

$$\dot{V}(x) = -x_1^2 + x_1 x_2^2 + a(-x_2^4 + x_2^3 u)$$

Apply the Young Inequalities:

$$x_1 x_2^2 \leq \frac{1}{2}x_1^2 + \frac{1}{2}x_2^4$$
$$x_2^3 u \leq \frac{\lambda^{4/3}}{4/3}x_2^4 + \frac{1}{4\lambda^4}u^4$$

Choose λ such that $\frac{\lambda^{4/3}}{4/3} = \frac{1}{2}$.

Example 3 (cont.)

Then

$$\dot{V}(x) \leq -\frac{1}{2}x_1^2 + \frac{1}{2}x_2^4 + a\left(-\frac{1}{2}x_2^4 + \frac{1}{4\lambda^4}u^4\right)$$

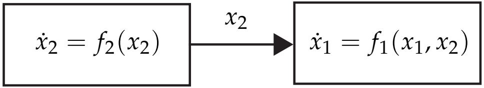
Let $a = 2$:

$$\dot{V}(x) \leq \underbrace{-\frac{1}{2}x_1^2 - \frac{1}{2}x_2^4}_{\leq -\alpha_3(|x|)} + \underbrace{\frac{1}{2\lambda^4}u^4}_{=\alpha_4(|u|)}$$

for an appropriate choice of α_3. Thus, the system is ISS.

Stability of Series Interconnections

$$\dot{x}_1 = f_1(x_1, x_2) \quad x_1 \in \mathbb{R}^{n_1}$$
$$\dot{x}_2 = f_2(x_2) \qquad x_2 \in \mathbb{R}^{n_2}$$



Suppose $x_2 = 0$ is globally asymptotically stable for $\dot{x}_2 = f_2(x_2)$ and $x_1 = 0$ is globally asymptotically stable for $\dot{x}_1 = f_1(x_1, 0)$. Is $(x_1, x_2) = 0$ globally asymptotically stable for the interconnection?

Answer: No.

Example 4

$$\dot{x}_1 = -x_1 + x_1^2 x_2$$
$$\dot{x}_2 = -x_2$$

exhibits finite time escape.

Stability of ISS Interconnections

Proposition: Consider the series interconnection:

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_2, u).$$

If the x_1 subsystem is ISS with x_2 viewed as an input, and the x_2 subsystem is ISS with input u, then the interconnection is ISS.

Example 3 Revisited

Example 3 revisited:

$$\dot{x}_1 = -x_1 + x_2^2 \quad \text{is ISS with respect to } x_2$$

$$\dot{x}_2 = -x_2 + u \quad \text{is ISS with input } u$$

$\Rightarrow$ the interconnection is ISS — an alternative to the proof in Ex. 3.

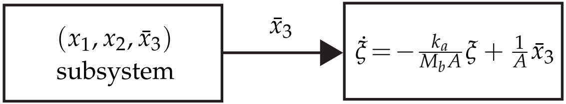
Stability of ISS Interconnections: GAS

Corollary: $(x_1, x_2) = 0$ is globally asymptotically stable when $u \equiv 0$.

Note that Example 4 fails the ISS condition for the x_1 subsystem.

Example

Example: Active suspension design example in Lecture 14:



The $(x_1, x_2, \bar{x}_3)$-subsystem globally asymptotically stabilized by backstepping. The ξ-subsystem is an asymptotically stable linear system, therefore ISS with respect to the input $\bar{x}_3$.