Lecture 14 – ME6402, Spring 2025
Backstepping

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Goals of Lecture 14

- Lyapunov-based design: Backstepping

Additional Reading

- Khalil 14.3
- Sastry 6.8

Lyapunov-based Feedback Design Examples

- Adaptive Control (last lecture)
- Backstepping (this lecture)
- Control Lyapunov Functions (later)

Backstepping

Feedback stabilization: Given the system

$$\dot{x} = f(x) + g(x)u$$

with input u, design a control law $u = \alpha(x)$ such that $x = 0$ is asymptotically stable for the closed-loop system:

$$\dot{x} = f(x) + g(x)\alpha(x).$$

Backstepping is a technique that simplifies this task for a class of systems.

- Khalil (Sec. 14.3), Sastry (Sec. 6.8)

Backstepping

Suppose a stabilizing feedback $u = \alpha(\eta)$, $\alpha(0) = 0$, is available for:

$$\dot{\eta} = F(\eta) + G(\eta)u \quad \eta \in \mathbb{R}^n, u \in \mathbb{R},\ F(0) = 0,$$

along with a Lyapunov function V such that

$$\frac{\partial V}{\partial \eta}\Big(F(\eta) + G(\eta)\alpha(\eta)\Big) \leq -W(\eta) < 0 \quad \forall \eta \neq 0.$$

Can we modify $\alpha(\eta)$ to stabilize the augmented system below?

$$\begin{aligned} \dot{\eta} &= F(\eta) + G(\eta)x \\ \dot{x} &= u. \end{aligned}$$

Backstepping

Define the error variable $\boxed{z = x - \alpha(\eta)}$ and change variables: $(\eta, x) \to (\eta, z)$:

$$\dot{\eta} = F(\eta) + G(\eta)\alpha(\eta) + G(\eta)z$$

$$\dot{z} = u - \dot{\alpha}(\eta, z)$$

where $\dot{\alpha}(\eta, z) = \dfrac{\partial \alpha}{\partial \eta} \cdot \Big(F(\eta) + G(\eta)\alpha(\eta) + G(\eta)z\Big)$. Take the new Lyapunov function:

$$V_+(\eta, z) = V(\eta) + \frac{1}{2}z^2.$$

$$\dot{V}_+ = \underbrace{\frac{\partial V}{\partial \eta}\Big(F(\eta) + G(\eta)\alpha(\eta)\Big)}_{\leq -W(\eta)} + \underbrace{\frac{\partial V}{\partial \eta}G(\eta)z + z(u - \dot{\alpha})}_{= z\left(u - \dot{\alpha} + \frac{\partial V}{\partial \eta}G(\eta)\right)}$$

$$\boxed{\begin{aligned}\dot{\eta} &= F(\eta) + G(\eta)x \\ \dot{x} &= u.\end{aligned}}$$

Backstepping

Let: $$u = \dot{\alpha} - \frac{\partial V}{\partial \eta} G(\eta) - kz, \quad k > 0.$$

Then, $\dot{V}_+ \leq -W(\eta) - kz^2 \Rightarrow (\eta, z) = 0$ is asymptotically stable.

$$\dot{\eta} = F(\eta) + G(\eta)x$$
$$\dot{x} = u.$$

$$\dot{V}_+ = \underbrace{\frac{\partial V}{\partial \eta}\Big(F(\eta) + G(\eta)\alpha(\eta)\Big)}_{\leq -W(\eta)}$$
$$+ \underbrace{\frac{\partial V}{\partial \eta} G(\eta) z + z(u - \dot{\alpha})}_{= z\left(u - \dot{\alpha} + \frac{\partial V}{\partial \eta} G(\eta)\right)}$$

Example

Example 1:

$$\dot{x}_1 = x_1^2 + x_2$$
$$\dot{x}_2 = u.$$

Treat x_2 as "virtual" control input for the x_1-subsystem:

$$\alpha(x_1) = -k_1 x_1 - x_1^2 \quad k_1 > 0$$
$$V_1(x_1) = \frac{1}{2} x_1^2.$$

Apply backstepping:

$$z_2 = x_2 - \alpha(x_1) = x_2 + k_1 x_1 + x_1^2$$
$$\dot{z}_2 = u - \dot{\alpha}$$
$$u = \dot{\alpha} - \frac{\partial V_1}{\partial x_1} - k_2 z_2, \quad k_2 > 0$$
$$= \underbrace{-(k_1 + 2x_1)(x_1^2 + x_2)}_{=\dot{\alpha}} - \underbrace{x_1}_{=\frac{\partial V_1}{\partial x_1}} - k_2 \underbrace{(x_2 + k_1 x_1 + x_1^2)}_{=z_2}.$$

- $$u = \dot{\alpha} - \frac{\partial V}{\partial \eta} G(\eta) - kz$$

Generalizing The Backstepping Idea

- Above we discussed backstepping over a pure integrator. The main idea generalizes trivially to:

$$\begin{aligned} \dot{\eta} &= F(\eta) + G(\eta)x \\ \dot{x} &= f(\eta, x) + g(\eta, x)u \end{aligned}$$

where $\eta \in \mathbb{R}^n$, $x \in \mathbb{R}$, and $g(\eta, x) \neq 0$ for all $(\eta, x) \in \mathbb{R}^{n+1}$.

Generalizing The Backstepping Idea

With the preliminary feedback

$$u = \frac{1}{g(\eta,x)}(-f(\eta,x)+v) \tag{1}$$

the x-subsystem becomes a pure integrator: $\dot{x} = v$. Substituting the backstepping control law from above:

$$v = \dot{\alpha} - \frac{\partial V}{\partial \eta}G(\eta) - kz, \quad z \triangleq x - \alpha(\eta), \quad k > 0$$

into (1), we get:

$$\boxed{u = \frac{1}{g(\eta,x)}\left(-f(\eta,x) + \dot{\alpha} - \frac{\partial V}{\partial \eta}G(\eta) - kz\right).}$$

Generalizing The Backstepping Idea

- Backstepping can be applied recursively to systems of the form:

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\
\dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)x_3 \\
\dot{x}_3 &= f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)x_4 \\
&\vdots \\
\dot{x}_n &= f_n(x) + g_n(x)u
\end{aligned}
$$

where $g_i(x_1, \ldots, x_i) \neq 0$ for all $x \in \mathbb{R}^n$, $i = 2, 3, \cdots, n$.

- Systems of this form are called "strict feedback systems."

Example

Example 2:

$$\dot{x}_1 = (x_1 x_2 - 1)x_1^3 + (x_1 x_2 + x_3^2 - 1)x_1$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = u.$$

Not in strict feedback form because x_3 appears too soon. In fact, this system is not globally stabilizable because the set $x_1 x_2 \geq 2$ is positively invariant regardless of u (next slide)

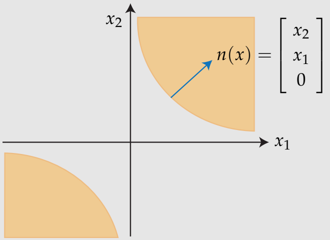
Example (cont.)

To see that the set $x_1x_2 \geq 2$ is positively invariant, note that

$$n(x)\cdot f(x,u) = [(x_1x_2-1)x_1^3 + (x_1x_2+x_3^2-1)x_1]x_2 + x_3x_1$$

and substitute $x_1x_2 = 2$:

$$\begin{aligned} &= \left(x_1^3 + (1+x_3^2)x_1\right)x_2 + x_3x_1 \\ &= \left(x_1^2 + (1+x_3^2)\right)x_1x_2 + x_3x_1 \\ &= 2x_1^2 + 2(1+x_3^2) + x_3x_1 \\ &= \underbrace{2x_1^2 + x_3x_1 + 2x_3^2}_{\geq 0} + 2 > 0. \end{aligned}$$



Example

Example 3:

$$\dot{x}_1 = x_1^2 x_2$$
$$\dot{x}_2 = u$$

Treat x_2 as virtual control and let $\boxed{\alpha_1(x_1) = -x_1}$ which stabilizes the x_1-subsystem, as verified with Lyapunov function $V_1(x_1) = \frac{1}{2}x_1^2$.
Then $z_2 := x_2 - \alpha_1(x_1)$ satisfies $\dot{z}_2 = u - \dot{\alpha}_1$, and

$$u = \dot{\alpha}_1 - \frac{\partial V_1}{\partial x_1} x_1^2 - k_2 z_2 = -x_1^2 x_2 - x_1^3 - k_2(x_2 + x_1)$$

achieves global asymptotic stability:

$$V = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2 \quad \Rightarrow \quad \dot{V} = -x_1{}^{4} - k_2 z_2^2.$$

Example (cont.)

Note that we can't conclude exponential stability due to the quartic term x_1^4 above (recall the Lyapunov sufficient condition for exponential stability in Lecture 11, p.2). In fact, the linearization of the closed-loop system proves the lack of exponential stability:

$$\begin{bmatrix} 0 & 0 \\ 0 & -k_2 \end{bmatrix} \rightarrow \lambda_{1,2} = 0, -k_2.$$

Design Example

Design example: Active suspension

$$M_b\ddot{x}_s = -k_a(x_s - x_a) - c_a(\dot{x}_s - \dot{x}_a)$$

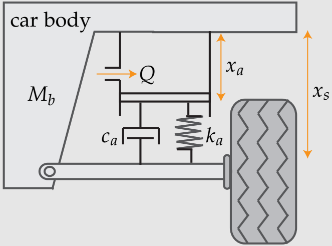
$$\dot{x}_a = \frac{1}{A}Q$$

A: effective piston surface

Flow: $\dot{Q} = -c_f Q + k_f u$

u: current applied to the solenoid valve (control input)

- Krstić et al., Nonlinear and Adaptive Control Design, Section 2.2.2.



Design Example (cont.)

Define state variables: $x_1 = x_s$, $x_2 = \dot{x}_s$, $x_3 = x_a$, $x_4 = Q$:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k_a}{M_b}(x_1 - x_3) - \frac{c_a}{M_b}\left(x_2 - \frac{1}{A}x_4\right)$$

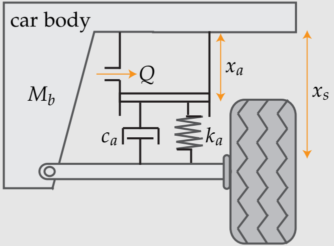
$$\dot{x}_3 = \frac{1}{A}x_4$$

$$\dot{x}_4 = -c_f x_4 + k_f u.$$

This system is not in strict feedback form due to the x_4 term in $\dot{x}_2$. To overcome this problem define:

$$\bar{x}_3 \triangleq \frac{k_a}{M_b}x_3 + \frac{c_a}{M_b A}x_4$$

$$\xi \triangleq x_3$$



$$M_b \ddot{x}_s = -k_a(x_s - x_a) - c_a(\dot{x}_s - \dot{x}_a)$$

$$\dot{x}_a = \frac{1}{A}Q$$

Flow: $\dot{Q} = -c_f Q + k_f u$

Design Example (cont.)

Change variables to $(x_1, x_2, \bar{x}_3, \xi)$:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{k_a}{M_b}x_1 - \frac{c_a}{M_b}x_2 + \bar{x}_3 \\
\dot{\bar{x}}_3 &= \frac{k_a - c_a c_f}{M_b A}x_4 + \frac{c_a k_f}{M_b A}u.
\end{aligned}$$

Two steps of backstepping starting with the virtual control law:

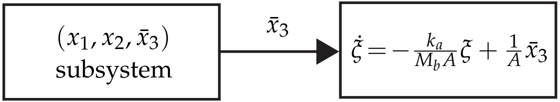
$$\alpha_1(x_1) = -c_1 x_1 - k_1 x_1^3$$

will stabilize the $(x_1, x_2, \bar{x}_3)$ subsystem.

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{k_a}{M_b}(x_1 - x_3) - \frac{c_a}{M_b}(x_2 - \frac{1}{A}x_4) \\
\dot{x}_3 &= \frac{1}{A}x_4 \\
\dot{x}_4 &= -c_f x_4 + k_f u. \\
\bar{x}_3 &\triangleq \frac{k_a}{M_b}x_3 + \frac{c_a}{M_b A}x_4 \\
\xi &\triangleq x_3
\end{aligned}$$

- The stiff nonlinearity $k_1 x_1^3$ prevents large excursions of x_1.

Full $(x_1, x_2, \bar{x}_3, \xi)$ system:



The ξ-subsystem is an asymptotically stable linear system driven by $\bar{x}_3$; therefore the full system is stabilized.