Lecture 13 – ME6402, Spring 2025

Time-Varying Systems and Lyapunov Design

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Goals of Lecture 13

- Linear Time-Varying Systems
- Differential Lyapunov Equation
- Lyapunov Design Examples

Additional Reading

- Khalil Chapter 4.6

Time-Varying Systems

$$\dot{x} = A(t)x \qquad x(t) = \Phi(t,t_0)x(t_0)$$

- The state transition matrix $\Phi(t,t_0)$ satisfies the equations:

$$\begin{aligned} \frac{\partial}{\partial t}\Phi(t,t_0) &= A(t)\Phi(t,t_0) \\ \frac{\partial}{\partial t_0}\Phi(t,t_0) &= -\Phi(t,t_0)A(t_0) \end{aligned}$$

- Khalil Section 4.6, Sastry Section 5.7

Linear Time-Varying Systems (cont.)

- No eigenvalue test for stability in the time-varying case:

$$A(t) = \begin{bmatrix} -1 + 1.5\cos^2 t & 1 - 1.5\sin t \cos t \\ -1 - 1.5\sin t \cos t & -1 + 1.5\sin^2 t \end{bmatrix}$$

eigenvalues: $-0.25 \mp i0.25\sqrt{7}$ for all t, but unstable:

$$\Phi(t,0) = \begin{bmatrix} e^{0.5t}\cos t & e^{-t}\sin t \\ e^{-0.5t}\sin t & e^{-t}\cos t \end{bmatrix}$$

- Khalil Section 4.6, Sastry Section 5.7

Linear Time-Varying Systems (cont.)

- For linear systems uniform asymptotic stability is equivalent to uniform exponential stability:
  Theorem: $x = 0$ is uniformly asymptotically stable if and only if
  $$\|\Phi(t,t_0)\| \leq ke^{-\lambda(t-t_0)} \quad \text{for some} \quad k > 0,\ \lambda > 0.$$
- Last lecture: $V(t,x) = x^T P(t) x$ proves uniform exp. stability if
  $$\begin{aligned} &\text{(i)} && \dot{P}(t) + A^T(t)P(t) + P(t)A(t) = -Q(t) \\ &\text{(ii)} && 0 < k_1 I \leq P(t) \leq k_2 I \\ &\text{(iii)} && 0 < k_3 I \leq Q(t) \text{ for all } t. \end{aligned}$$

- Khalil Thm. 4.11, Sastry Thm. 5.33

Linear Time-Varying Systems (cont.)

The converse is also true:

Theorem: Suppose $x = 0$ is uniformly exponentially stable, $A(t)$ is continuous and bounded, $Q(t)$ is continuous and symmetric, and there exist $k_3, k_4 > 0$ such that

$$0 < k_3 I \leq Q(t) \leq k_4 I \quad \text{for all } t.$$

Then, there exists a symmetric $P(t)$ satisfying (i)–(ii):

$$\text{(i)}\ \dot{P}(t) + A^T(t)P(t) + P(t)A(t) = -Q(t)$$

$$\text{(ii)}\ 0 < k_1 I \leq P(t) \leq k_2 I$$

- For stable linear systems, there always exists quadratic Lyapunov functions.
- Find them by choosing any positive definite $Q(t)$ and solve (differential) Lyapunov equation.

Proof:

Time-invariant: $P = \int_0^\infty e^{A^T\tau} Q e^{A\tau} d\tau$

Time-varying: $P(t) = \int_t^\infty \Phi^T(\tau,t) Q(\tau) \Phi(\tau,t) d\tau$

Using the Leibniz rule, property (2), and $\Phi(t,t) = I$ we obtain:

$$\begin{aligned}
\dot{P}(t) = \int_t^\infty & \left(\frac{\partial}{\partial t} \Phi^T(\tau,t) Q(\tau) \Phi(\tau,t) + \Phi^T(\tau,t) Q(\tau) \frac{\partial}{\partial t} \Phi(\tau,t) \right) d\tau \\
& - \Phi^T(t,t) Q(t) \Phi(t,t) \\
= \int_t^\infty & \left(-A^T(t) \Phi^T(\tau,t) Q(\tau) \Phi(\tau,t) - \Phi^T(\tau,t) Q(\tau) \Phi(\tau,t) A(t) \right) d\tau \\
& - \Phi^T(t,t) Q(t) \Phi(t,t) \\
= -A^T(t) & P(t) - P(t) A(t) - Q(t).
\end{aligned}$$

Lyapunov-based Feedback Design Examples

- Adaptive Control (this lecture)
- Backstepping (next lecture)
- Control Lyapunov Functions (later)

Adaptive Control with an Unknown Parameter

Consider

$$\dot{y} = a^* y + u.$$

Goal: Stabilize the origin even when a^* is unknown.

- Can this be achieved with linear feedback, $u = -Ky$ for some K?
- Can this be achieved with static nonlinear feedback, $u = k(y)$?

Adaptive Control with an Unknown Parameter

Consider

$$\dot{y} = a^* y + u.$$

Goal: Stabilize the origin even when a^* is unknown.

- Can this be achieved with linear feedback, $u = -Ky$ for some K?

  A: Not unless an a priori bound on $|a^*|$ is known.
- Can this be achieved with static nonlinear feedback, $u = k(y)$?

  A: Try $u = -ky^3$, $k > 0$. If $a^* > 0$, this introduces two new, stable equilibria at $\pm\sqrt{a^*/k}$. Trajectories, therefore remain bounded, but still no stability at origin.

Adaptive Control with an Unknown Parameter, Dynamic Feedback

Let's try building a dynamic feedback controller.

- Dynamic means the controller itself has a state variable, and therefore memory.
- Let the controller estimate a with $\hat{a}$.

Goal: Design update law for $\dot{a}$ and controller $u(y, \hat{a})$ to stabilize origin.

- Our approach is Lyapunov-based: we choose a Lyapunov function candidate and work to make it an actual Lyapunov function

- System:

$$\dot{y} = a^* y + u,$$

a^* unknown.

Adaptive Control with an Unknown Parameter, Dynamic Feedback (cont.)

$$\dot{y} = a^* y + u(y, \hat{a}), \qquad \dot{\hat{a}} = ??$$

State variables are y, $\hat{a}$. Try

$$V(y, \hat{a}) = \frac{1}{2} y^2 + \frac{1}{2}(\hat{a} - a^*)^2$$

Adaptive Control with an Unknown Parameter, Dynamic Feedback (cont.)

$$\dot{y} = a^* y + u(y, \hat{a}), \qquad \dot{\hat{a}} = ??$$

State variables are y, $\hat{a}$. Try

$$V(y, \hat{a}) = \frac{1}{2} y^2 + \frac{1}{2} (\hat{a} - a^*)^2$$

$$\implies \dot{V} = y\dot{y} + (\hat{a} - a^*)\dot{\hat{a}} = y(a^* y + u) + (\hat{a} - a^*)\dot{\hat{a}}$$

$$= a^*(y^2 - \dot{\hat{a}}) + uy + \hat{a}\dot{\hat{a}}$$

- Want $\dot{V} \leq -y^2$ (for example) so that $y \to 0$ by LaSalle
- u and $\dot{\hat{a}}$ can be functions of $\hat{a}$ and y, but **not** a^*
- Therefore, we choose $\dot{\hat{a}} = y^2$ and then $\dot{V} = uy + \hat{a} y^2$. Choose

$$u(y, \hat{a}) = -\hat{a} y - y$$

Adaptive Control with an Unknown Parameter, Dynamic Feedback (cont.)

Final controlled system:

$$\dot{y} = a^* y + u(y, \hat{a}) = (a^* - \hat{a} - 1)y$$
$$\dot{\hat{a}} = y^2$$

Lyapunov function $V(y, \hat{a}) = \frac{1}{2}y^2 + \frac{1}{2}(\hat{a} - a^*)^2$ gives $\dot{V} = -y^2$.
Apply LaSalle.

- Does $y \to 0$?
- Does $\hat{a} \to a^*$?

Adaptive Control with an Unknown Parameter, Dynamic Feedback (cont.)

Final controlled system:

$$\dot{y} = a^* y + u(y, \hat{a}) = (a^* - \hat{a} - 1)y$$
$$\dot{\hat{a}} = y^2$$

Lyapunov function $V(y, \hat{a}) = \frac{1}{2}y^2 + \frac{1}{2}(\hat{a} - a^*)^2$ gives $\dot{V} = -y^2$. Apply LaSalle.

- Does $y \to 0$?

  A: Yes

- Does $\hat{a} \to a^*$?

  A: Not necessarily. For example, $(0, \hat{a})$ is an equilibrium for any $\hat{a}$. We achieve asymptotic stability of system, but not necessarily estimator convergence

Model Reference Adaptive Control

Illustrated on same first order system:

$$\dot{y} = a^* y + u$$

where a^* is unknown.

Reference model:

$$\dot{y}_m = -a y_m + r(t) \quad a > 0,\ r(t) : \text{reference signal.}$$

Goal: Design a controller that guarantees $y(t) - y_m(t) \to 0$ without the knowledge of a^*.

Model Reference Adaptive Control (cont.)

If we knew a^*, we would choose:

$$u = -\underbrace{(a^* + a)}_{=:k^*}y + r(t) \quad \Rightarrow \quad \dot{y} = -ay + r(t).$$

The tracking error $\boxed{e(t) := y(t) - y_m(t)}$ then satisfies:

$$\dot{e} = -ae \ \Rightarrow \ e(t) \to 0 \text{ exponentially.}$$

Adaptive design when a^* (therefore, k^*) is unknown:

$$u = -k(t)y + r(t)$$

where $\dot{k}(t)$ is to be designed. Then:

$$\dot{e} = \dot{y} - \dot{y}_m = a^*y - k(t)y + ay_m = -ae - \underbrace{(k(t) - k^*)}_{=:\tilde{k}(t)}y$$

where adding and subtracting ay gives the final equality.

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$$\dot{y} = a^*y + u$$

$$\dot{y}_m = -ay_m + r(t) \ a > 0,$$

$r(t)$: reference signal.

Model Reference Adaptive Control (cont.)

Use the Lyapunov function: $V = \frac{1}{2}e^2 + \frac{1}{2}\tilde{k}^2$:

$$\dot{V} = -ae^2 - \tilde{k}ey + \tilde{k}\dot{\tilde{k}}$$
$$= -ae^2 + \tilde{k}(\dot{\tilde{k}} - ey).$$

Note $\dot{\tilde{k}} = \dot{k}$ and choose $\boxed{\dot{k} = ey}$ so that $\dot{V} = -ae^2$.

This guarantees stability of $(e, \tilde{k}) = (0,0)$ and boundedness of $(e(t), \tilde{k}(t))$ since the level sets of $V = \frac{1}{2}e^2 + \frac{1}{2}\tilde{k}^2$ are positively invariant. In addition, if $r(t)$ is bounded, then $y_m(t)$ in (15) is bounded, and so is $y(t) = y_m(t) + e(t)$.

- $$\dot{y}_m = -ay_m + r(t) \;\; a > 0,$$

  $r(t)$: reference signal.

Model Reference Adaptive Control (cont.)

Then we can apply the Theorem from Lecture 12, page 5, to the time-varying model

$$\dot{e} = -ae - y(t)\tilde{k}, \quad \dot{\tilde{k}} = y(t)e,$$

and conclude from $\dot{V} = -ae^2$ that $e(t) \to 0$.

Whether $\tilde{k}(t) \to 0$ ($k(t) \to k^*$) depends on further properties of the reference signal $r(\cdot)$ that are beyond the scope of this lecture.

- $$\dot{y}_m = -ay_m + r(t) \;\; a > 0,$$

  $r(t)$: reference signal.