Lecture 12 – ME6402, Spring 2025
Time-Varying Systems Continued

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Goals of Lecture 12

- Lyapunov theory in time-varying systems

Additional Reading

- Khalil Chapter 4.6, 8.3

Comparison Functions

Definition: A continuous function $\alpha : [0,\infty) \to [0,\infty)$ is class-$\mathcal{K}$ if it is zero at zero and strictly increasing. It is class-$\mathcal{K}_\infty$ if, in addition, $\alpha(r) \to \infty$ as $r \to \infty$.

A continuous function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is class-$\mathcal{KL}$ if:

1. $\beta(\cdot, s)$ is class-$\mathcal{K}$ for every fixed s,
2. $\beta(r, \cdot)$ is decreasing and $\beta(r,s) \to 0$ as $s \to \infty$, for every fixed r.

Example: $\alpha(r) = \tan^{-1}(r)$ is class-$\mathcal{K}$, $\alpha(r) = r^c, c > 0$ is class-$\mathcal{K}_\infty$, $\beta(r,s) = r^c e^{-s}$ is class-$\mathcal{KL}$.

Comparison Functions

Proposition: If $V(\cdot)$ is positive definite, then we can find class-$\mathcal{K}$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|).$$

If $V(\cdot)$ is radially unbounded, we can choose $\alpha_1(\cdot)$ to be class-$\mathcal{K}_\infty$.

Example: $V(x) = x^T P x \quad P = P^T > 0$

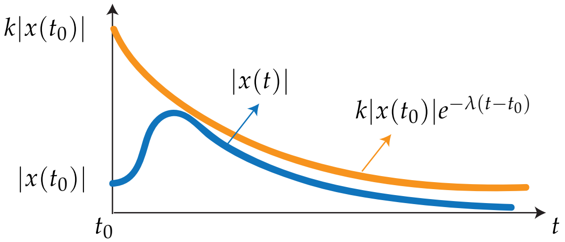
$$\alpha_1(|x|) = \lambda_{\min}(P)|x|^2 \quad \alpha_2(|x|) = \lambda_{\max}(P)|x|^2.$$

Stability Definitions

- $x = 0$ is uniformly stable if there exists a class-$\mathcal{K}$ function $\alpha(\cdot)$ and a constant $c > 0$ such that
$$|x(t)| \leq \alpha(|x(t_0)|)$$
for all $t \geq t_0$ and for every initial condition such that $|x(t_0)| \leq c$.
- uniformly asymptotically stable if there exists a class-$\mathcal{KL}$ $\beta(\cdot,\cdot)$ s.t.
$$|x(t)| \leq \beta(|x(t_0)|, t - t_0)$$
for all $t \geq t_0$ and for every initial condition such that $|x(t_0)| \leq c$.
- globally uniformly asymptotically stable if $c = \infty$.
- uniformly exponentially stable if $\beta(r,s) = kre^{-\lambda s}$ for some $k, \lambda > 0$:
$$|x(t)| \leq k|x(t_0)|e^{-\lambda(t-t_0)}$$
for all $t \geq t_0$ and for every initial condition such that $|x(t_0)| \leq c$.

Stability of Time-Varying Systems

$k > 1$ allows for overshoot:



- uniformly exponentially stable if $\beta(r,s) = kre^{-\lambda s}$ for some $k, \lambda > 0$:

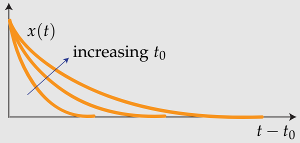
$$|x(t)| \leq k|x(t_0)|e^{-\lambda(t-t_0)}$$

for all $t \geq t_0$ and for every initial condition such that $|x(t_0)| \leq c$.

Example 1

Example: Consider the following system, defined for $t > -1$:

$$\dot{x} = \frac{-x}{1+t}$$



Example 1

Example: Consider the following system, defined for $t > -1$:

$$\dot{x} = \frac{-x}{1+t}$$

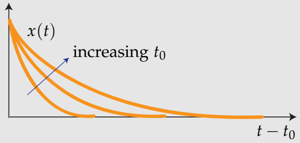
$$x(t) = x(t_0)e^{\int_{t_0}^{t} \frac{-1}{1+s} ds} = x(t_0)e^{\log(1+s)|_t^{t_0}}$$

$$= x(t_0)e^{\log \frac{1+t_0}{1+t}} = x(t_0)\frac{1+t_0}{1+t}$$

$|x(t)| \leq |x(t_0)| \implies$ the origin is uniformly stable with $\alpha(r) = r$.

The origin is also asymptotically stable, but not uniformly, because the convergence rate depends on t_0:

$$x(t) = x(t_0)\frac{1+t_0}{1+t_0+(t-t_0)} = \frac{x(t_0)}{1+\frac{t-t_0}{1+t_0}}.$$



Example 2

Example:

$$\dot{x} = -x^3 \quad \Rightarrow \quad x(t) = \mathrm{sgn}(x(t_0))\sqrt{\frac{x_0^2}{1+2(t-t_0)x_0^2}}$$

$x = 0$ is asymptotically stable but not exponentially stable because $1/\sqrt{t}$ decays more slowly than any exponential.

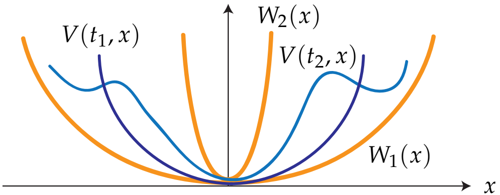
Exponential Stability

Proposition: $x = 0$ is exponentially stable for $\dot{x} = f(x)$, $f(0) = 0$, if and only if $A \triangleq \left.\frac{\partial f}{\partial x}\right|_{x=0}$ is Hurwitz, that is $\Re\lambda_i(A) < 0 \ \forall i$.

Although strict inequality in $\Re\lambda_i(A) < 0$ is not necessary for asymptotic stability (see example above where $A = 0$), it *is* necessary for exponential stability.

Lyapunov's Stability Theorem for Time-Varying Systems

1. If $W_1(x) \leq V(t,x) \leq W_2(x)$ and $\dot{V}(t,x) \triangleq \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq 0$ for some positive definite functions $W_1(\cdot)$, $W_2(\cdot)$ on a domain D that includes the origin, then $x = 0$ is uniformly stable.

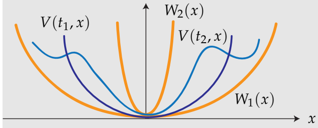


- Khalil, Section 4.5

Lyapunov's Stability Theorem for Time-Varying Systems

1. If $W_1(x) \leq V(t,x) \leq W_2(x)$ and $\dot{V}(t,x) \triangleq \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq 0$ for some positive definite functions $W_1(\cdot)$, $W_2(\cdot)$ on a domain D that includes the origin, then $x = 0$ is uniformly stable.
2. If, further, $\dot{V}(t,x) \leq -W_3(x)$ $\forall x \in D$ for some positive definite $W_3(\cdot)$, then $x = 0$ is uniformly asymptotically stable.
3. If $D = \mathbb{R}^n$ and $W_1(\cdot)$ is radially unbounded, then $x = 0$ is globally uniformly asymptotically stable.
4. If $W_i(x) = k_i |x|^a$, $i = 1, 2, 3$, for some constants $k_1, k_2, k_3, a > 0$, then $x = 0$ is uniformly exponentially stable.

- Khalil, Section 4.5



Lyapunov's Stability Theorem for Time-Varying Systems

Proof:

1. $\alpha_1(|x|) \le W_1(x) \le V(t,x) \le W_2(x) \le \alpha_2(|x|)$

$$
\begin{aligned}
\dot V \le 0 &\Rightarrow V(x(t),t) \le V(x(t_0),t_0) \\
&\Rightarrow \alpha_1(|x(t)|) \le \alpha_2(|x(t_0)|) \\
&\Rightarrow |x(t)| \le \alpha(|x(t_0)|) \triangleq (\alpha_1^{-1} \circ \alpha_2)(|x(t_0)|).
\end{aligned}
$$

Note: The inverse of a class-$\mathcal{K}$ function is well defined locally (globally if $\mathcal{K}_\infty$) and is class-$\mathcal{K}$. The composition of two class-$\mathcal{K}$ functions is also class-$\mathcal{K}$.

1. If $W_1(x) \le V(t,x) \le W_2(x)$ and $\dot V(t,x) \triangleq \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le 0$ for some positive definite functions $W_1(\cdot)$, $W_2(\cdot)$ on a domain D that includes the origin, then $x = 0$ is uniformly stable.
2. If, further, $\dot V(t,x) \le -W_3(x)\ \forall x \in D$ for some positive definite $W_3(\cdot)$, then $x = 0$ is uniformly asymptotically stable.
3. If $D = \mathbb{R}^n$ and $W_1(\cdot)$ is radially unbounded, then $x = 0$ is globally uniformly asymptotically stable.
4. If $W_i(x) = k_i |x|^a$, $i = 1,2,3$, for some constants $k_1, k_2, k_3, a > 0$, then $x = 0$ is uniformly exponentially stable.

Lyapunov's Stability Theorem for Time-Varying Systems

Proof:

2. $\dot{V} \le -W_3(x) \le -\alpha_3(|x|) \le -\alpha_3(\alpha_2^{-1}(V)) \triangleq -\gamma(V)$

$$\frac{d}{dt}V(t,x(t)) \le -\gamma(V(t,x(t)))$$

Let $y(t)$ be the solution of $\dot{y} = -\gamma(y)$, $y(t_0) = V(t_0,x(t_0))$. Then,

$$V(t,x(t)) \le y(t).$$

Since $\dot{y} = -\gamma(y)$ is a first order differential equation and $-\gamma(y) < 0$ when $y > 0$, we conclude monotone convergence of $y(t)$ to 0:

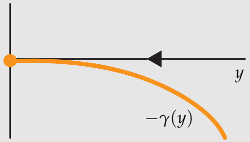
$$y(t) = \beta(y(t_0), t - t_0) \implies V(t,x(t)) \le \beta(\underbrace{V(t_0,x(t_0))}_{\le \alpha_2(|x(t_0)|)}, t - t_0)$$

$$\Rightarrow \alpha_1(|x(t)|) \le \beta(\alpha_2(|x(t_0)|), t - t_0)$$

$$\Rightarrow |x(t)| \le \tilde{\beta}(|x(t_0)|, t - t_0)$$

$$\triangleq \alpha_1^{-1}(\beta(\alpha_2(|x(t_0)|), t - t_0))$$

1. If $W_1(x) \le V(t,x) \le W_2(x)$ and $\dot{V}(t,x) \triangleq \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le 0$ for some positive definite functions $W_1(\cdot)$, $W_2(\cdot)$ on a domain D that includes the origin, then $x = 0$ is uniformly stable.
2. If, further, $\dot{V}(t,x) \le -W_3(x)$ $\forall x \in D$ for some positive definite $W_3(\cdot)$, then $x = 0$ is uniformly asymptotically stable.
3. If $D = \mathbb{R}^n$ and $W_1(\cdot)$ is radially unbounded, then $x = 0$ is globally uniformly asymptotically stable.
4. If $W_i(x) = k_i|x|^a$, $i = 1,2,3$, for some constants $k_1, k_2, k_3, a > 0$, then $x = 0$ is uniformly exponentially stable.



Lyapunov's Stability Theorem for Time-Varying Systems

Proof:

3. If $\alpha_1(\cdot)$ is class $\mathcal{K}_\infty$ then $\alpha_1^{-1}(\cdot)$ exists globally above.
4. $\alpha_3(|x|) = k_3|x|^a$, $\alpha_2(|x|) = k_2|x|^a$

$$\Rightarrow \gamma(V) = \alpha_3(\alpha_2^{-1}(V)) = k_3\left(\left(\frac{V}{k_2}\right)^{\frac{1}{a}}\right)^a = \frac{k_3}{k_2}V$$

$$\dot{y} = -\frac{k_3}{k_2}y \;\Rightarrow\; y(t) = y(t_0)e^{-(k_2/k_2)(t-t_0)}$$

$$\beta(r,s) = re^{-(k_3/k_2)s} \Rightarrow \tilde{\beta}(r,s) = \left(\frac{k_2}{k_1}r^a e^{-(k_3/k_2)s}\right)^{\frac{1}{a}} = \left(\frac{k_2}{k_1}\right)^{\frac{1}{a}} re^{-\frac{k_3 a}{k_2}s}.$$

1. If $W_1(x) \le V(t,x) \le W_2(x)$ and $\dot{V}(t,x) \triangleq \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t,x) \le 0$ for some positive definite functions $W_1(\cdot)$, $W_2(\cdot)$ on a domain D that includes the origin, then $x=0$ is uniformly stable.
2. If, further, $\dot{V}(t,x) \le -W_3(x)$ $\forall x \in D$ for some positive definite $W_3(\cdot)$, then $x=0$ is uniformly asymptotically stable.
3. If $D = \mathbb{R}^n$ and $W_1(\cdot)$ is radially unbounded, then $x=0$ is globally uniformly asymptotically stable.
4. If $W_i(x) = k_i|x|^a$, $i = 1,2,3$, for some constants $k_1, k_2, k_3, a > 0$, then $x=0$ is uniformly exponentially stable.

Example 1

Example:

$$\dot{x} = -g(t)x^3 \quad \text{where} \quad g(t) \geq 1 \quad \text{for all} \quad t$$

$$V(x) = \frac{1}{2}x^2 \quad \Rightarrow \quad \dot{V}(t,x) = -g(t)x^4 \leq -x^4 \triangleq W_3(x)$$

Globally uniformly asymptotically stable but not exponentially stable. Take $g(t) \equiv 1$ as a special case.

Example 2

Example: $\dot{x} = A(t)x$. Take $V(x) = x^T P(t)x$:

$$\begin{aligned}\dot{V}(x) &= x^T \dot{P}(t)x + \dot{x}^T P(t)x + x^T P(t)\dot{x} \\ &= x^T \underbrace{(\dot{P} + A^T P + PA)}_{\triangleq -Q(t)} x\end{aligned}$$

If $k_1 I \leq P(t) \leq k_2 I$ and $k_3 I \leq Q(t)$, $k_1, k_2, k_3 > 0$, then

$$k_1|x|^2 \leq V(t,x) \leq k_2|x|^2 \quad \text{and} \quad \dot{V}(t,x) \leq -k_3|x|^2$$

$\Rightarrow$ global uniform exponential stability.

A Lasalle-Krasovskii-Type Result

What if $W_3(\cdot)$ is only semidefinite?

Lasalle-Krasovskii Invariance Principle is not applicable to time-varying systems. Instead, use the following (weaker) result:

Theorem: Suppose $W_1(x) \leq V(t,x) \leq W_2(x)$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W_3(x),$$

where $W_1(\cdot), W_2(\cdot)$ are positive definite and $W_3(\cdot)$ is positive semidefinite. Suppose, further, $W_1(\cdot)$ is radially unbounded, $f(t,x)$ is locally Lipschitz in x and bounded in t, and $W_3(\cdot)$ is C^1. Then

$$\boxed{W_3(x(t)) \to 0 \text{ as } t \to \infty.}$$

Note: This proves convergence to $S = \{x : W_3(x) = 0\}$ whereas the Invariance Principle, when applicable, guarantees convergence to the largest invariant set within S.

- Khalil, Section 8.3

Example 3

Example:

$$\begin{aligned}\dot{x}_1 &= -x_1 + w(t)x_2 \\ \dot{x}_2 &= -w(t)x_1\end{aligned}$$

$V(t,x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \;\Rightarrow\; \dot{V}(t,x) = -x_1^2$. If $w(t)$ is bounded in t then the theorem above implies $x_1(t) \to 0$ as $t \to \infty$, but no guarantee about the convergence of $x_2(t)$ to zero.

By contrast, if $w(t) \equiv w \neq 0$, then we can use the Invariance Principle and conclude $x_2(t) \to 0$ (show this).

Barbalat's Lemma

Barbalat's Lemma (used in proving the theorem above):

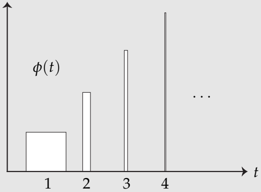
If $\lim_{t\to\infty}\int_0^t \phi(\tau)d\tau$ exists and is finite, and $\phi(\cdot)$ is *uniformly continuous* then $\phi(t)\to 0$ as $t\to\infty$.

Uniform continuity in Barbalat's Lemma can't be relaxed:

Example: Let $\phi(t)$ be a sequence of pulses centered at $k = 1,2,3,\ldots$ with amplitude $= k$, width $= 1/k^3$, then

$$\int_0^\infty \phi(t)dt = \sum_{k=1}^{\infty}\frac{1}{k^2} < \infty \quad \text{but} \quad \phi(t)\not\to 0.$$

- Uniformly continuous means: For every $\varepsilon > 0$ there exists $\delta > 0$ such that $\forall t_1, t_2\ |t_1 - t_2| \le \delta \Rightarrow |\phi(t_1) - \phi(t_2)| \le \varepsilon$. Boundedness of the derivative $\dot\phi(t)$ implies uniform continuity.



A Lasalle-Krasovskii-Type Result: Proof

Proof of the theorem:

$$\alpha_1(|x|) \leq V(t,x) \leq \alpha_2(|x|) \quad \alpha_1 \in \mathcal{K}_\infty$$

$$\Rightarrow |x(t)| \leq \alpha_1^{-1}(\alpha_2(|x(t_0)|))$$

$x(t)$ bounded $\Rightarrow \dot{x}(t) = f(t,x(t))$ is bounded $\Rightarrow x(t)$ is uniformly continuous.

$$\dot{V}(t,x) \leq -W_3(x(t))$$

$$\Rightarrow V(x(T)) - V(x(t_0),t_0) \leq -\int_{t_0}^{T} W_3(x(t))dt$$

$$\Rightarrow \int_{t_0}^{\infty} W_3(x(t))dt \leq V(x(t_0),t_0) < \infty.$$

Since $W_3(\cdot)$ is C^1, it is uniformly continuous on the bounded domain where $x(t)$ resides. So, by Barbalat's Lemma, $W_3(x(t)) \to 0$ as $t \to \infty$.

- Theorem: Suppose $W_1(x) \leq V(t,x) \leq W_2(x)$

  $$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W_3(x),$$

  where $W_1(\cdot), W_2(\cdot)$ are positive definite and $W_3(\cdot)$ is positive semidefinite. Suppose, further, $W_1(\cdot)$ is radially unbounded, $f(t,x)$ is locally Lipschitz in x and bounded in t, and $W_3(\cdot)$ is C^1. Then

  $W_3(x(t)) \to 0$ as $t \to \infty$.