Lecture 11 – ME6402, Spring 2025
Lyapunov's Linearization Method

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Goals of Lecture 11

- Further tools for studying systems based on their linearization
- Define region of attraction
- Obtain Lyapunov estimates of the region of attraction
- Introduce time-varying systems and comparison functions

Additional Reading

- Khalil Chapter 4.3-4.7

Lyapunov's Linearization Method

$$\dot{x} = f(x) \quad f(0) = 0$$

Define $A = \left.\dfrac{\partial f(x)}{\partial x}\right|_{x=0}$ and decompose $f(x)$ as

$$f(x) = Ax + g(x) \quad \text{where} \quad \frac{|g(x)|}{|x|} \to 0 \text{ as } |x| \to 0.$$

Theorem: The origin is asymptotically stable if $\Re\{\lambda_i(A)\} < 0$ for each eigenvalue, and unstable if $\Re\{\lambda_i(A)\} > 0$ for some eigenvalue.

Note: We can conclude only *local* asymptotic stability from this linearization. Inconclusive if A has eigenvalues on the imaginary axis.

Lyapunov's Linearization Method (cont.)

Proof: Find $P = P^T > 0$ such that $A^T P + PA = -Q < 0$. Use $V(x) = x^T P x$ as a Lyapunov function for the nonlinear system $\dot{x} = Ax + g(x)$.

$$\dot{V}(x) =$$

- Theorem: The origin is asymptotically stable if $\Re\{\lambda_i(A)\} < 0$ for each eigenvalue, and unstable if $\Re\{\lambda_i(A)\} > 0$ for some eigenvalue.

Lyapunov's Linearization Method (cont.)

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$$\begin{aligned}\dot{V}(x) &= x^T P(Ax + g(x)) + (Ax + g(x))^T P x \\ &= x^T (PA + A^T P)x + 2x^T P g(x) \\ &\leq -x^T Q x + 2|x|\|P\||g(x)|\end{aligned}$$

$$\lambda_{\min}(Q)|x|^2 \leq x^T Q x \leq \lambda_{\max}(Q)|x|^2$$

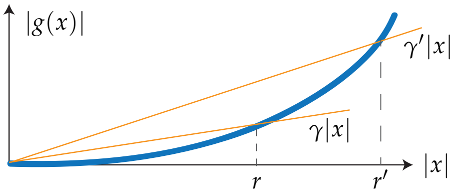
$$\dot{V}(x) \leq -\lambda_{\min}(Q)|x|^2 + 2\|P\||x||g(x)|$$

- Theorem: The origin is asymptotically stable if $\Re\{\lambda_i(A)\} < 0$ for each eigenvalue, and unstable if $\Re\{\lambda_i(A)\} > 0$ for some eigenvalue.

Lyapunov's Linearization Method (cont.)

Proof (cont.):

Since $\frac{|g(x)|}{|x|} \to 0$ as $x \to 0$, for any $\gamma > 0$ we can find $r > 0$ such that
$|x| \leq r \Rightarrow |g(x)| \leq \gamma |x|$; see the illustration below for the case $x \in \mathbb{R}$.



Thus, $|x| \leq r(\gamma) \;\Rightarrow\; \dot{V}(x) \leq -\lambda_{\min}(Q)|x|^2 + 2\gamma \|P\| |x|^2$.

- Theorem: The origin is asymptotically stable if $\Re\{\lambda_i(A)\} < 0$ for each eigenvalue, and unstable if $\Re\{\lambda_i(A)\} > 0$ for some eigenvalue.
- $\dot{V}(x) \leq -\lambda_{\min}(Q)|x|^2 + 2\|P\||x||g(x)|$

Lyapunov's Linearization Method (cont.)

Proof (cont.):

Choose $\gamma < \dfrac{\lambda_{\min}(Q)}{2\|P\|}$ so that $\dot{V}$ is negative definite in a ball of radius $r(\gamma)$ around the origin, and appeal to Lyapunov's Stability Theorem (Lecture 8) to conclude (local) asymptotic stability.

- Theorem: The origin is asymptotically stable if $\Re\{\lambda_i(A)\} < 0$ for each eigenvalue, and unstable if $\Re\{\lambda_i(A)\} > 0$ for some eigenvalue.
- $\dot{V}(x) \leq -\lambda_{\min}(Q)|x|^2 + 2\|P\||x||g(x)|$
- $|x| \leq r(\gamma) \;\Rightarrow\; \dot{V}(x) \leq -\lambda_{\min}(Q)|x|^2 + 2\gamma\|P\||x|^2$

Region of Attraction

$$R_A = \{x : \phi(t,x) \to 0\}$$

"Quantifies" local asymptotic stability. Global asymptotic stability: $R_A = \mathbb{R}^n$.

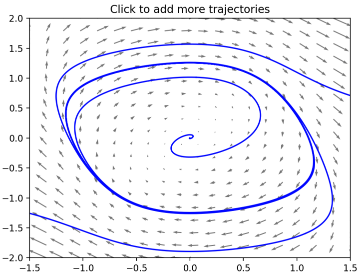
Proposition: If $x = 0$ is asymptotically stable, then its region of attraction is an open, connected, invariant set. Moreover, the boundary is formed by trajectories.

Region of Attraction

Example: van der Pol system in reverse time:

$$\dot{x}_1 = -x_2$$
$$\dot{x}_2 = x_1 - x_2 + x_2^3$$

The boundary is the (unstable) limit cycle. Trajectories starting within the limit cycle converge to the origin.

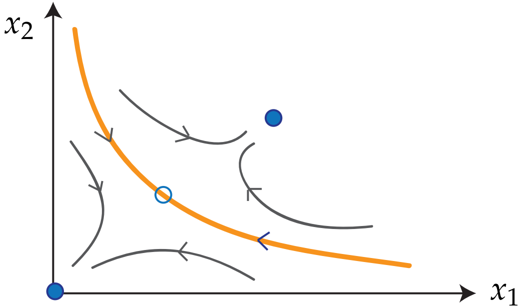


Region of Attraction

Example: bistable switch:

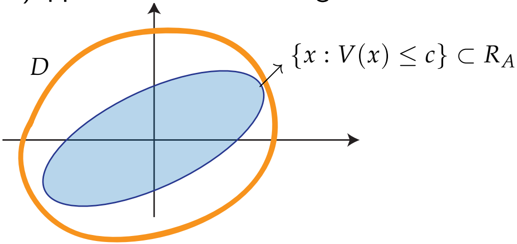
$$\dot{x}_1 = -ax_1 + x_2$$

$$\dot{x}_2 = \frac{x_1^2}{1+x_1^2} - bx_2$$



Estimating the Region of Attraction with a Lyapunov Function

Suppose $\dot{V}(x) < 0$ in $D - \{0\}$. The level sets of V inside D are invariant and trajectories starting in them converge to the origin. Therefore we can use the largest levet set of V that fits into D as an (under)approximation of the region of attraction.



This estimate depends on the choice of Lyapunov function. A simple (but often conservative) choice is: $V(x) = x^T P x$ where P is selected for the linearization (see p.1).

Time-Varying Systems

$$\dot{x} = f(t, x) \quad f(t, 0) \equiv 0$$

To simplify the definitions of stability and asymptotic stability for the equilibrium $x = 0$, we first define a class of functions known as "comparison functions."

- Khalil (Sec. 4.5), Sastry (Sec. 5.2)

Comparison Functions

Definition: A continuous function $\alpha : [0,\infty) \to [0,\infty)$ is class-$\mathcal{K}$ if it is zero at zero and strictly increasing. It is class-$\mathcal{K}_\infty$ if, in addition, $\alpha(r) \to \infty$ as $r \to \infty$.

A continuous function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is class-$\mathcal{KL}$ if:

1. $\beta(\cdot, s)$ is class-$\mathcal{K}$ for every fixed s,
2. $\beta(r, \cdot)$ is decreasing and $\beta(r,s) \to 0$ as $s \to \infty$, for every fixed r.

Example: $\alpha(r) = \tan^{-1}(r)$ is class-$\mathcal{K}$, $\alpha(r) = r^c, c > 0$ is class-$\mathcal{K}_\infty$, $\beta(r,s) = r^c e^{-s}$ is class-$\mathcal{KL}$.

Comparison Functions

Proposition: If $V(\cdot)$ is positive definite, then we can find class-$\mathcal{K}$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|).$$

If $V(\cdot)$ is radially unbounded, we can choose $\alpha_1(\cdot)$ to be class-$\mathcal{K}_\infty$.

Example: $V(x) = x^T P x \quad P = P^T > 0$

$$\alpha_1(|x|) = \lambda_{\min}(P)|x|^2 \quad \alpha_2(|x|) = \lambda_{\max}(P)|x|^2.$$

Stability Definitions

Definition: $x = 0$ is stable if for every $\varepsilon > 0$ and t_0, there exists $\delta > 0$ such that

$$|x(t_0)| \leq \delta(t_0, \varepsilon) \implies |x(t)| \leq \varepsilon \quad \forall t \geq t_0.$$

If the same δ works for all t_0, *i.e.* $\delta = \delta(\varepsilon)$, then $x = 0$ is uniformly stable.

It is easier to define uniform stability and uniform asymptotic stability using comparison functions (next slide)

Stability Definitions

- $x = 0$ is uniformly stable if there exists a class-$\mathcal{K}$ function $\alpha(\cdot)$ and a constant $c > 0$ such that
$$|x(t)| \leq \alpha(|x(t_0)|)$$
for all $t \geq t_0$ and for every initial condition such that $|x(t_0)| \leq c$.
- uniformly asymptotically stable if there exists a class-$\mathcal{KL}$ $\beta(\cdot,\cdot)$ s.t.
$$|x(t)| \leq \beta(|x(t_0)|, t - t_0)$$
for all $t \geq t_0$ and for every initial condition such that $|x(t_0)| \leq c$.
- globally uniformly asymptotically stable if $c = \infty$.
- uniformly exponentially stable if $\beta(r,s) = kre^{-\lambda s}$ for some $k, \lambda > 0$:
$$|x(t)| \leq k|x(t_0)|e^{-\lambda(t-t_0)}$$
for all $t \geq t_0$ and for every initial condition such that $|x(t_0)| \leq c$.