# Lecture 10 – ME6402, Spring 2025
## *LaSalle-Krasovskii Invariance Principle*

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Goals of Lecture 10

- LaSalle-Krasovskii Invariance Principle, applicable when $\dot{V}(x) \leq 0$
- Lyapunov functions for linear systems

Additional Reading

- Khalil Chapter 4.2-4.3

# Lyapunov's Stability Theorem– Recap from Lecture 9

1. Let $D$ be an open, connected subset of $\mathbb{R}^n$ that includes $x = 0$. If there exists a $C^1$ function $V : D \to \mathbb{R}$ such that

   $V(0) = 0$ and $V(x) > 0 \quad \forall x \in D - \{0\} \qquad$ (positive definite)

   and

   $\dot{V}(x) := \nabla V(x)^T f(x) \leq 0 \quad \forall x \in D \quad$ (negative semidefinite)

   then $x = 0$ is stable.
2. If $\dot{V}(x) < 0 \quad \forall x \in D - \{0\} \qquad$ (negative definite)

   then $x = 0$ is asymptotically stable.
3. If, in addition, $D = \mathbb{R}^n$ and

   $$|x| \to \infty \implies V(x) \to \infty \qquad \text{(radially unbounded)}$$

   then $x = 0$ is globally asymptotically stable.

# LaSalle-Krasovskii Invariance Principle

- Applicable to time-invariant systems.
- Allows us to conclude asymptotic stability from $\dot{V}(x) \leq 0$ if additional conditions hold:

Suppose $\Omega_c = \{x : V(x) \leq c\}$ is bounded and $\dot{V}(x) \leq 0$ in $\Omega_c$. Define $S = \{x \in \Omega_c : \dot{V}(x) = 0\}$ and let $M$ be the largest invariant set in $S$. Then, for every $x(0) \in \Omega_c$, $x(t) \to M$.

Corollary: If no solution other than $x(t) \equiv 0$ can stay identically in $S$ then $M = \{0\}$ and we conclude asymptotic stability.

# LaSalle-Krasovskii Invariance Principle (Example)

Example (from last lecture):

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -ax_2 - g(x_1) \quad a > 0,\ xg(x) > 0 \ \ \forall x \neq 0$$

$$V(x) = \int_0^{x_1} g(y)dy + \frac{1}{2}x_2^2 \quad \Longrightarrow \quad \dot{V}(x) = -ax_2^2$$

$$S = \{x \in \Omega_c | x_2 = 0\}$$

If $x(t)$ stays identically in $S$, then $x_2(t) \equiv 0 \Longrightarrow \dot{x}_2(t) \equiv 0 \Longrightarrow g(x_1(t)) \equiv 0 \Longrightarrow x_1(t) \equiv 0 \Longrightarrow$ asymptotic stability from Corollary.

# LaSalle-Krasovskii Invariance Principle (Example 2)

Example (linear system): Same system as before with $g(x_1) = bx_1$:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -ax_2 - bx_1 \quad a > 0, b > 0$$

$V(x) = \frac{b}{2}x_1^2 + \frac{1}{2}x_2^2 \implies \dot{V}(x) = -ax_2^2 \implies$ Invariance Principle works as in the example above.

# LaSalle-Krasovskii Invariance Principle (Example 2 cont.)

Alternatively, construct another Lyapunov function with negative definite $\dot{V}(x)$. Try $V(x) = x^T P x$ where $P = P^T > 0$ is to be selected.

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -a x_2 - b x_1 \quad a > 0, b > 0$$

# LaSalle-Krasovskii Invariance Principle (Example 2 cont.)

Alternatively, construct another Lyapunov function with negative definite $\dot{V}(x)$. Try $V(x) = x^T P x$ where $P = P^T > 0$ is to be selected.

$$\dot{V}(x) = x^T P \dot{x} + \dot{P} x = x^T (A^T P + PA) x \text{ where } A = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}$$

Let $P = \frac{1}{2}\begin{bmatrix} b & \varepsilon \\ \varepsilon & 1 \end{bmatrix}$, that is $V(x) = \frac{b}{2}x_1^2 + \varepsilon x_1 x_2 + \frac{1}{2}x_2^2$.

Note that $P > 0$ if $\varepsilon^2 < b$.

$$A^T P + PA = \begin{bmatrix} -\varepsilon b & -\varepsilon a/2 \\ -\varepsilon a/2 & \varepsilon - a \end{bmatrix}$$

$\leq 0$ if $\varepsilon = 0$

$< 0$ if

$$\underbrace{0 < \varepsilon < a \text{ and } \varepsilon b(a - \varepsilon) > \frac{\varepsilon^2 a^2}{4}}_{0<\varepsilon<\frac{ba}{2}}$$

$\dot{x}_1 = x_2$

$\dot{x}_2 = -a x_2 - b x_1 \quad a > 0, b > 0$

# Linear Systems

$$\dot{x} = Ax \quad x \in \mathbb{R}^n$$

$x = 0$ is stable if $\Re\{\lambda_i(A)\} \leq 0$ for all $i = 1, \cdots, n$ and eigenvalues on the imaginary axis have Jordan blocks of order one.[a] It is asymptotically stable if $\Re\{\lambda_i(A)\} < 0$ for all $i$, *i.e.*, $A$ is "Hurwitz."

[a] *i.e.*, if $\lambda$ is an eigenvalue of multiplicity $q$ then $\lambda I - A$ must have rank $n - q$.

- Sastry (Sec. 5.7-5.8), Khalil (Sec. 4.3)

# Linear Systems Example

Example:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$ is unstable: $$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 0 \end{aligned}\right\} x_1(t) = x_1(0) + x_2(0)t$$

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$ is stable.

# Lyapunov Functions for Linear Systems

$$V(x) = x^T P x \qquad P = P^T > 0$$
$$\dot{V}(x) = x^T (A^T P + PA) x$$

If $\exists P = P^T > 0$ such that $A^T P + PA = -Q < 0$, then $A$ is Hurwitz.

The converse is also true (next slide)

# Lyapunov Functions for Linear Systems (cont.)

Theorem: $A$ is Hurwitz if and only if for any $Q = Q^T > 0$, there exists $P = P^T > 0$ such that

$$A^T P + PA = -Q. \qquad (1)$$

Moreover, the solution $P$ is unique.

Proof:

(if) From analysis above ((2) at right), the Lyapunov function $V(x) = x^T P x$ proves asymptotic stability which means $A$ is Hurwitz.

- (1) Is known as the Lyapunov Equation. The Matlab command `lyap(A',Q)` returns the solution $P$.
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$$V(x) = x^T P x \quad P = P^T > 0$$
$$\dot{V}(x) = x^T (A^T P + PA) x \qquad (2)$$

# Lyapunov Functions for Linear Systems (cont.)

Proof (cont.):

(only if) Assume $\Re\{\lambda_i(A)\} < 0\ \forall i$. Show $\exists P = P^T > 0$ such that $A^T P + PA = -Q$.

Candidate:

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt.$$

- The integral exists because $\|e^{At}\| \leq \kappa e^{-\alpha t}$.
- $P = P^T$
- $P > 0$ because $x^T P x = \int_0^\infty (e^{At}x)^T Q \underbrace{(e^{At}x)}_{\triangleq \phi(t,x)} dt \geq 0$ and

  $x^T P x = 0 \implies \phi(t,x) \equiv 0 \implies x = 0$ because $e^{At}$ is nonsingular.

- Theorem: $A$ is Hurwitz if and only if for any $Q = Q^T > 0$, there exists $P = P^T > 0$ such that

$$A^T P + PA = -Q.$$

# Lyapunov Functions for Linear Systems (cont.)

Proof (cont.):

(only if cont.) Assume $\Re\{\lambda_i(A)\} < 0 \ \forall i$. Show $\exists P = P^T > 0$ such that $A^T P + PA = -Q$.

Candidate (cont.):

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt.$$

- $A^T P + PA = \int_0^\infty \underbrace{\left(A^T e^{A^T t} Q e^{At} + e^{A^T t} Q e^{At} A\right)}_{= \frac{d}{dt}\left(e^{At} Q e^{At}\right)} dt$

$$= e^{A^T t} Q e^{At} \Big|_0^\infty = 0 - Q = -Q$$

- Theorem: $A$ is Hurwitz if and only if for any $Q = Q^T > 0$, there exists $P = P^T > 0$ such that

$$A^T P + PA = -Q.$$

# Lyapunov Functions for Linear Systems (cont.)

Proof (cont.):

(only if cont.) Uniqueness:

Suppose there is another $\hat{P} = \hat{P}^T > 0$ satisfying $\hat{P} \neq P$, and $A^T\hat{P} + \hat{P}A = -Q$.

Then, $(P - \hat{P})A + A^T(P - \hat{P}) = 0$. Define $W(x) = x^T(P - \hat{P})x$.

Now,

$$\frac{d}{dt}W(x(t)) = 0 \implies W(x(t)) = W(x(0)) \quad \forall t.$$

Since $A$ is Hurwitz, $x(t) \to 0$ and $W(x(t)) \to 0$.

Combining the two statements above, we conclude $W(x(0)) = 0$ for any $x(0)$. This is possible only if $P - \hat{P} = 0$ which contradicts $\hat{P} \neq P$.

- Theorem: $A$ is Hurwitz if and only if for any $Q = Q^T > 0$, there exists $P = P^T > 0$ such that

$$A^T P + PA = -Q.$$

# Invariance Principle Applied to Linear Systems

$$A^T P + PA = -Q \leq 0$$

Can we conclude that $A$ is Hurwitz if $Q$ is only semidefinite?

Decompose $Q$ as $Q = C^T C$ where $C \in \mathbb{R}^{r \times n}$, $r$ is the rank of $Q$.

$$\dot{V}(x) = -x^T Q x = -x^T C^T C x = -y^T y$$

where $y \triangleq Cx$. The invariance principle guarantees asymptotic stability if

$$y(t) = Cx(t) \equiv 0 \implies x(t) \equiv 0.$$

This implications is true if the pair $(C, A)$ is observable.

# Invariance Principle Applied to Linear Systems Example

Example (beginning of the lecture):

$$A = \begin{bmatrix} 0 & 1 \\ -b & a \end{bmatrix} \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} \quad \Longrightarrow \quad C = [0 \quad \sqrt{a}]$$

$(C, A)$ is observable if $b \neq 0$.