Lecture 1 – ME6402, Spring 2025

Nonlinear Control Systems: A Brief Introduction

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Goals of Lecture 1

- Introduce nonlinear systems
- Define equilibria, linearization, stability in scalar systems
- Provide some canonical examples

Additional Reading

- Khalil, Chapter 1
- Sastry, Chapter 1

Nonlinear Systems

$$\dot{x} = Ax + Bu \quad \longrightarrow \quad \dot{x} = f(x,u)$$

- Analysis:

$\dot{x} = f(x)$	$f : \mathbb{R}^n \to \mathbb{R}^n$	time-invariant (autonomous)
$\dot{x} = f(t,x)$	$f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$	time-varying (non-autonomous)

- Design:

$$\dot{x} = f(x,u) \quad u \text{ to be designed as a function of } x.$$

- We use the shorthand notation $\dot{x} = f(x)$ for $\frac{d}{dt}x(t) = f(x(t))$.

Equilibria

$x = x^*$ is an equilibrium for $\dot{x} = f(x)$ if $f(x^*) = 0$.

Example: Linear system $\dot{x} = Ax$.

If A is nonsingular, $x^* = 0$ is the unique equilibrium.

If A is singular, the nullspace defines a continuum of equilibria.

A nonsingular matrix A has the following equivalent properties:

- $\det(A) \neq 0$
- The rows (or columns) are linearly independent

The nullspace of A is:

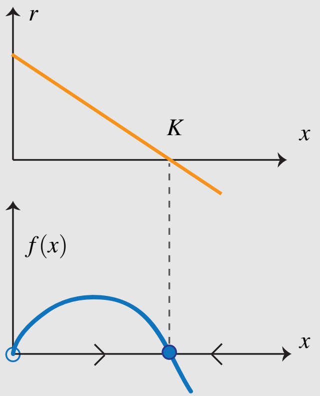
- $\text{Nullspace}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$

Example: Logistic Growth

<u>Example:</u> *Logistic growth model* in population dynamics

$$\dot{x} = f(x) = \underbrace{r\left(1 - \frac{x}{K}\right)}_{\text{growth rate}} x, \qquad r > 0, \quad K > 0$$

- $x > 0$ denotes the population (at time t)
- r is the intrinsic growth rate
- K is called the carrying capacity (the maximum population size that the environment can sustain)



Determining Stability in Scalar Systems

For systems with a scalar state variable $x \in \mathbb{R}$, stability can be determined from the sign of $f(x)$ around the equilibrium. In this example $f(x) > 0$ for $x \in (0, K)$, and $f(x) < 0$ for $x > K$; therefore

$$\begin{aligned} x &= 0 \quad \text{unstable equilibrium} \\ x &= K \quad \text{asymptotically stable.} \end{aligned}$$

- *Logistic growth model* in population dynamics

$$\dot{x} = f(x) = \underbrace{r\left(1 - \frac{x}{K}\right)}_{\text{growth rate}} x,$$

$$r > 0, \quad K > 0$$

Linearization

Local stability properties of x^* can be determined by linearizing the vector field $f(x)$ at x^*:

$$f(x^* + \tilde{x}) = \underbrace{f(x^*)}_{=\,0} + \underbrace{\left.\frac{\partial f}{\partial x}\right|_{x=x^*}}_{\triangleq\, A} \tilde{x} + \text{higher order terms}$$

Thus, the linearized model is:

$$\dot{\tilde{x}} = A\tilde{x}.$$

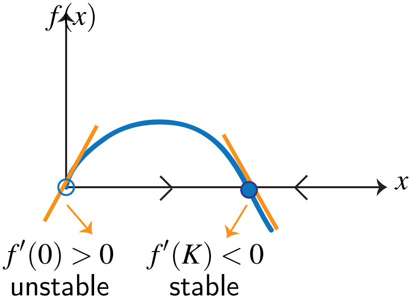
If $\Re\lambda_i(A) < 0$ for each eigenvalue λ_i of A, then x^* is asymp. stable.

If $\Re\lambda_i(A) > 0$ for some eigenvalue λ_i of A, then x^* is unstable.

- the expansion is a first-order Taylor series approximation around the linearized state $\tilde{x} = x - x^*$
- $\Re\lambda_i(A) = 0$ indicates marginal stability or oscillatory behavior for the linearized system. E.g. an undamped pendulum

Example (continued)

Example: Logistic growth model above:



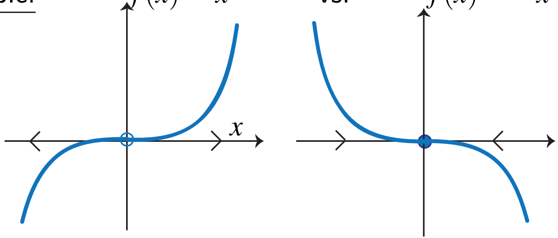
Caveats

1. Only local properties can be determined from the linearization.
   Example: The logistic growth model linearized at $x = 0$ ($\dot{x} = rx$) would incorrectly predict unbounded growth of $x(t)$. In reality, $x(t) \to K$.

Caveats (continued)

2. If $\Re\lambda_i(A) \leq 0$ with equality for some i, then linearization is inconclusive as a stability test. Higher order terms determine stability.

   Example: $f(x) = x^3$ vs. $f(x) = -x^3$



$f'(0) = 0$ in each case, but one is stable and the other is unstable.

Second order example: Pendulum

$$\ell m\ddot{\theta} = -k\ell\dot{\theta} - mg\sin\theta$$

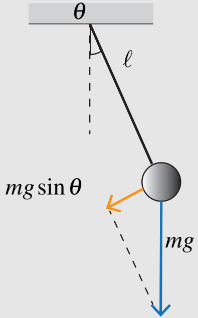
Define $x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$. State space: $S^1 \times \mathbb{R}$.

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k}{m}x_2 - \frac{g}{\ell}\sin x_1$$

Equilibria: $(0,0)$ and $(\pi,0)$

$$\frac{\partial f}{\partial x} =$$



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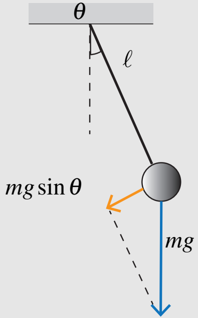
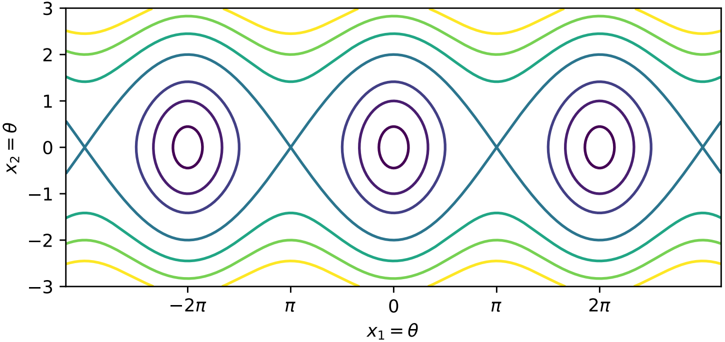
Equilibria: $(0,0)$ and $(\pi,0)$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos x_1 & -\frac{k}{m} \end{bmatrix} = \begin{cases} \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{k}{m} \end{bmatrix} & \text{(stable) at } x_1 = 0 \\ \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & -\frac{k}{m} \end{bmatrix} & \text{(unstable) at } x_1 = \pi \end{cases}$$



Phase portrait: plot of $x_1(t)$ vs. $x_2(t)$ for 2nd order systems



Phase portrait of the pendulum for the undamped case $k = 0$.