# *ME 6402 – Lecture 9* [1]

## LASALLE-KRASOVSKII INVARIANCE PRINCIPLE

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Overview:

- LaSalle-Krasovskii Invariance Principle, applicable when $\dot{V}(x) \leq 0$.
- Lyapunov functions for linear systems

Additional Reading:

- Khalil, Chapter 4.2-4.3

## *Recall*

Recall from the end of Lecture 8 the following example:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_2 - g(x_1) \quad a \geq 0,\ xg(x) > 0\ \ \forall x \in (-b, c) - \{0\} \end{aligned}$$

The pendulum is a special case with $g(x) = \sin(x)$.

We considered the candidate Lyapunov function:

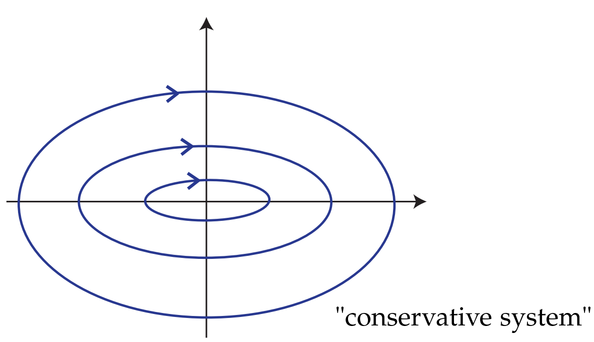
$$V(x) = \int_0^{x_1} g(y)dy + \frac{1}{2}x_2^2$$

which resulted in the derivative condition on the interval $D = (-b, c) - \{0\}$:

$$\dot{V}(x) = -ax_2^2$$

Since $\dot{V}(x)$ is negative semidefinite $\implies$ stable.

If $a = 0$, no asymptotic stability because $\dot{V}(x) = 0 \implies V(x(t)) = V(x(0))$.



If $a > 0$, the system is asymptotically stable but the Lyapunov function above doesn't allow us to reach that conclusion. This is because $\dot{V}(x) = 0$ on the line $x_2 = 0$. We need either another $V$ with negative definite $\dot{V}$, or the Lasalle-Krasovskii Invariance Principle.

## *LaSalle-Krasovskii Invariance Principle*

- Applicable to time-invariant systems.
- Allows us to conclude asymptotic stability from $\dot{V}(x) \leq 0$ if additional conditions hold.

**Theorem**: LaSalle Invariance Principle. *Let $\Omega \subset D$ be a compact set that is positively invariant with respect to the system $\dot{x} = f(x)$. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(x) \leq 0$ in $\Omega$. Let $E$ be the set of all points in $\Omega$ where $\dot{V}(x) = 0$. Let $M$ be the largest invariant set in $E$. Then every solution starting in $\Omega$ approaches $M$ as $t \to \infty$.*

**Corollary**: Lasalle-Krasovskii Invariance Principle[2]. *Let $x = 0$ be an equilibrium point for the system $\dot{x} = f(x)$. Let $V : D \to \mathbb{R}$ be a continuously differentiable positive definite function on a domain $D$ containing the origin $x = 0$, such that $\dot{V}(x) \leq 0$ in $D$. Let $S = \{x \in D \text{ s.t. } \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in $S$, other than the trivial solution $x(t) \equiv 0$. Then, the origin is asymptotically stable.*

[2] Also known as the theorems of Barbashin and Krasovskii, who proved it before the introduction of LaSalle's invariance principle

- Note: practically, the set $D$ is often selected to be the level set $\Omega_c = \{x : V(x) \leq c\}$ which is bounded such that $\dot{V}(x) \leq 0$ in $\Omega_c$. Then, we define $S = \{x \in \Omega_c : \dot{V}(x) = 0\}$ and let $M$ be the largest invariant set in $S$. Then, for every $x(0) \in \Omega_c$, $x(t) \to M$.
- If no solution other than $x(t) \equiv 0$ can stay identically in $S$ then $M = \{0\}$ and we conclude asymptotic stability.

**Corollary**: Lasalle-Krasovskii Invariance Principle for Globally Asymptotic Stability. *Let $x = 0$ be an equilibrium point for the system $\dot{x} = f(x)$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable, radially unbounded, positive definite function such that $\dot{V}(x) \leq 0$ for all $x \in \mathbb{R}^n$. Let $S = \{x \in \mathbb{R}^n \text{ s.t. } \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in $S$, other than the trivial solution $x(t) \equiv 0$. Then, the origin is globally asymptotically stable.*

Example (continued from before):

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_2 - g(x_1) \quad a > 0,\; xg(x) > 0 \;\; \forall x \neq 0 \end{aligned} \tag{1}$$

$$V(x) = \int_0^{x_1} g(y)dy + \frac{1}{2}x_2^2 \quad \Longrightarrow \quad \dot{V}(x) = -ax_2^2$$
$$S = \{x \in \Omega_c | x_2 = 0\}$$

If $x(t)$ stays identically in $S$, then $x_2(t) \equiv 0 \Longrightarrow \dot{x}_2(t) \equiv 0 \Longrightarrow g(x_1(t)) \equiv 0 \Longrightarrow x_1(t) \equiv 0 \Longrightarrow$ asymptotic stability from Corollary.

Example (linear system): Same system above with $g(x_1) = bx_1$:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_2 - bx_1 \quad a > 0, b > 0 \end{aligned} \tag{2}$$

$V(x) = \frac{b}{2}x_1^2 + \frac{1}{2}x_2^2 \Longrightarrow \dot{V}(x) = -ax_2^2 \Longrightarrow$ Invariance Principle works as in the example above.

Alternatively, construct another Lyapunov function with negative definite $\dot{V}(x)$. Try $V(x) = x^T P x$ where $P = P^T > 0$ is to be selected.

$$\dot{V}(x) = x^T P \dot{x} + \dot{P}x = x^T(A^T P + PA)x \text{ where } A = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}$$

Then, if we select $P$ to satisfy $PA + A^T P = -Q$ for some positive definite symmetric matrix $Q = Q^T > 0$, then

$$\dot{V}(x) = -x^T Q x < 0$$

and we can conclude that the origin is asymptotically stable.

This method uses what's known as the *Lyapunov Equation*, we will explore this further next.

## *Linear Systems*

Sastry (Sec. 5.7-5.8), Khalil (Sec. 4.3)

The linear time-invariant system

$$\dot{x} = Ax \quad x \in \mathbb{R}^n \tag{3}$$

has an equilibrium point at the origin ($x = 0$). From linear system theory, we know that the equilibrium point is stable if and only if $\Re\{\lambda_i(A)\} \leq 0$ for all $i = 1, \cdots, n$ *and* eigenvalues on the imaginary axis have Jordan blocks of order one.[3]

Example:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \lambda_{1,2} = 0, \ \text{rank}(\lambda I - A) = 1 \Longrightarrow \text{unstable}$$
$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \lambda_{1,2} = 0, \ \text{rank}(\lambda I - A) = 0 \Longrightarrow \text{stable}$$

[3] *i.e.*, if $\lambda$ is an eigenvalue of multiplicity $q$ then $\lambda I - A$ must have rank $n - q$. This is Theorem 4.5 in Khalil

When all eigenvalues of $A$ satisfy $\Re\lambda_i < 0$, $A$ is said to be *Hurwitz*. The origin is asymptotically stable if and only if $A$ is Hurwitz.

As alluded to before, asymptotic stability of the origin can also be investigated using Lyapunov's method.

## *Lyapunov Functions for Linear Systems*

$$\begin{aligned} V(x) &= x^T P x \qquad P = P^T > 0 \\ \dot{V}(x) &= x^T(A^T P + PA)x \end{aligned} \tag{4}$$

If $\exists P = P^T > 0$ such that $A^T P + PA = -Q < 0$, then $A$ is Hurwitz. The converse is also true:

<u>Theorem:</u> $A$ is Hurwitz if and only if for any $Q = Q^T > 0$, there exists $P = P^T > 0$ such that

$$A^T P + PA = -Q. \tag{5}$$

Moreover, the solution $P$ is unique.

(5) is known as the Lyapunov Equation. The Matlab command `lyap(A',Q)` returns the solution $P$.

<u>Proof:</u>

(if) From (4) above, the Lyapunov function $V(x) = x^T P x$ proves asymptotic stability which means $A$ is Hurwitz.

(only if) Assume $\Re\{\lambda_i(A)\} < 0\ \forall i$. Show $\exists P = P^T > 0$ such that $A^T P + PA = -Q$.

Candidate:

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt. \tag{6}$$

- The integral exists because the integrand is a sum of terms[4] of the form $t^{k-1}\exp(\lambda_i t)$, where $\Re\lambda_i < 0$. So $\|e^{At}\| \le \kappa e^{-\alpha t}$.
- $P = P^T$
- $P > 0$ because $x^T P x = \int_0^\infty (e^{At}x)^T Q \underbrace{(e^{At}x)}_{\triangleq \phi(t,x)} dt \ge 0$ and

  $x^T P x = 0 \Longrightarrow \phi(t,x) \equiv 0 \Longrightarrow x = 0$ because $e^{At}$ is nonsingular.
- $A^T P + PA = \int_0^\infty \underbrace{\left(A^T e^{A^T t} Q e^{At} + e^{A^T t} Q e^{At} A\right)}_{= \frac{d}{dt}\left(e^{A^T t} Q e^{At}\right)} dt$

  $= e^{A^T t} Q e^{At}\Big|_0^\infty = 0 - Q = -Q$

Uniqueness:

Suppose there is another $\hat{P} = \hat{P}^T > 0$ satisfying $\hat{P} \neq P$, and $A^T \hat{P} + \hat{P} A = -Q$.

[4] This comes from the Jordan form $J = P^{-1}AP$ which leads to:

$$\begin{aligned} \exp(At) &= P\exp(Jt)P^{-1} \\ &= \sum_{i=1}^{r}\sum_{k=1}^{m} t^{k-1}\exp(\lambda_i t) R_{ik} \end{aligned}$$

with $r$ being the number of Jordan blocks, and $m_i$ being the order of the Jordan block $J_i$.

$\Longrightarrow (P - \hat{P})A + A^T(P - \hat{P}) = 0$

Define $W(x) = x^T(P - \hat{P})x$.

$\frac{d}{dt}W(x(t)) = 0 \Longrightarrow W(x(t)) = W(x(0)) \quad \forall t.$

Since $A$ is Hurwitz, $x(t) \to 0$ and $W(x(t)) \to 0$.

Combining the two statements above, we conclude $W(x(0)) = 0$ for any $x(0)$. This is possible only if $P - \hat{P} = 0$ which contradicts $\hat{P} \neq P$.

## *Invariance Principle Applied to Linear Systems*

Similar to the nonlinear case, we can relax the positive definiteness requirement on $Q$ for proving asymptotic stability of linear systems. I.e., the Lyapunov equation can be satisfied for:

$$A^T P + PA = -Q \leq 0$$

In other words, we conclude that $A$ is Hurwitz if $Q$ is only semidefinite?

Sketch Proof: Decompose $Q$ as $Q = C^T C$ where $C \in \mathbb{R}^{r \times n}$, $r$ is the rank of $Q$.

$$\dot{V}(x) = -x^T Q x = -x^T C^T C x = -y^T y$$

where $y \triangleq Cx$. The invariance principle guarantees asymptotic stability if

$$y(t) = Cx(t) \equiv 0 \implies x(t) \equiv 0.$$

This implication is true if the pair $(C, A)$ is observable[5] since observability implies that the only state $x$ that produces identically zero output $y(t)$ for all time is $x \equiv 0$.

[5] A pair $(C, A)$ is observable if the observability matrix

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has full rank, i.e., $\text{rank}(\mathcal{O}) = n$.

Example (beginning of the lecture):

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_2 - bx_1 \quad a > 0, b > 0 \end{aligned}$$

Which can be rewritten in the form:

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}}_{A} x$$

If we selected the $Q$ matrix

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix},$$

then $Q$ is positive semidefinite. However, we can use the invariance principle above by selecting $C$ satisfying $C^T C = Q$:

$$C = \begin{bmatrix} 0 & \sqrt{a} \end{bmatrix}$$

and observing that $(C, A)$ is observable if $b \neq 0$:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{a} \\ -\sqrt{a}b & -\sqrt{a}a \end{bmatrix} \implies \text{rank}(\mathcal{O}) = 2 \text{ if } b \neq 0$$

*Solving the Lyapunov Equation*

Assume we are given the system $\dot{x} = Ax$ with

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

Assume we are asked to solve the Lyapunov equation with $Q = I$. One method of solving the Lyapunov equation is to rearrange it in the form $Mx = y$ with $x$ and $y$ defined by stacking the elements of $P$ and $Q$.

Let

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

The Lyapunov equation $A^T P + PA = -Q$ can be written as

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} p_{12} & p_{22} \\ -p_{11} - p_{12} & -p_{12} - p_{22} \end{bmatrix} + \begin{bmatrix} p_{12} & -p_{11} - p_{12} \\ p_{22} & -p_{12} - p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 2p_{12} & -p_{11} - p_{12} + p_{22} \\ -p_{11} - p_{12} + p_{22} & -2p_{12} - 2p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Putting this all together:

$$\begin{bmatrix} 0 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{22} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

This yields the solution

$$\begin{bmatrix} p_{11} \\ p_{12} \\ p_{22} \end{bmatrix} = \begin{bmatrix} 1.5 \\ -0.5 \\ 1.0 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1.0 \end{bmatrix}$$