ME 6402 – Lecture 8 [1]

LYAPUNOV STABILITY THEORY

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[1] Based on notes created by Murat Arcak and licensed under a

Overview:

- Define Lyapunov stability notions
- Lyapunov Stability Theorems

Additional Reading:

- Khalil, Chapter 4
- Sastry, Chapter 5

Lyapunov Stability Theory

Khalil Chapter 4, Sastry Chapter 5

Consider a time invariant system

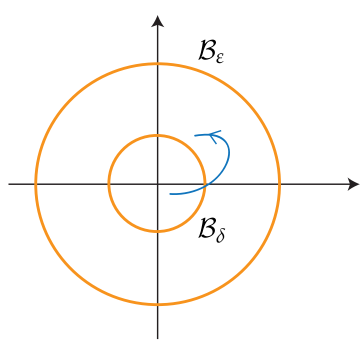
$$\dot{x} = f(x)$$

and assume equilibrium at $x = 0$, *i.e.* $f(0) = 0$. If the equilibrium of interest is $x^* \neq 0$, let $\tilde{x} = x - x^*$:

$$\dot{\tilde{x}} = f(x) = f(\tilde{x} + x^*) \triangleq \tilde{f}(\tilde{x}) \implies \tilde{f}(0) = 0.$$

Definition: The equilibrium $x = 0$ is stable if for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x(0)| \leq \delta \implies |x(t)| \leq \varepsilon \quad \forall t \geq 0. \tag{1}$$

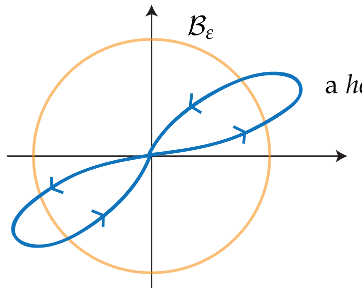


It is unstable if not stable.

Asymptotically stable if stable and $x(t) \to 0$ for all $x(0)$ in a neighborhood of $x = 0$.

Globally asymptotically stable if stable and $x(t) \to 0$ for every $x(0)$.

Note that $x(t) \to 0$ does not necessarily imply stability: one can construct an example where trajectories converge to the origin, but only after a large detour that violates the stability definition.




Aleksandr Lyapunov (1857-1918)

Lyapunov's Stability Theorem

1. Let D be an open, connected subset of $\mathbb{R}^n$ that includes $x = 0$. If there exists a C^1 function $V : D \to \mathbb{R}$ such that

$$V(0) = 0 \text{ and } V(x) > 0 \quad \forall x \in D - \{0\} \qquad \text{(positive definite)}$$

   and

$$\dot{V}(x) := \nabla V(x)^T f(x) \leq 0 \quad \forall x \in D \qquad \text{(negative semidefinite)}$$

   then $x = 0$ is stable.

2. If $\dot{V}(x) < 0 \quad \forall x \in D - \{0\}$ (negative definite) then $x = 0$ is asymptotically stable.

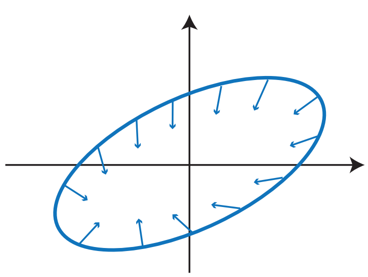
3. If, in addition, $D = \mathbb{R}^n$ and

$$|x| \to \infty \implies V(x) \to \infty \qquad \text{(radially unbounded)}$$

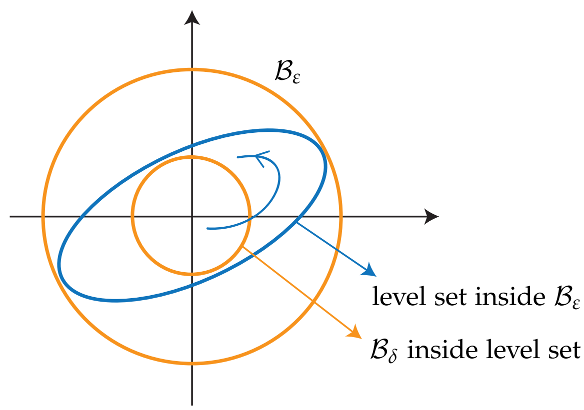
   then $x = 0$ is globally asymptotically stable.

Sketch of the proof:

The sets $\Omega_c \triangleq \{x : V(x) \leq c\}$ for constants c are called *level sets* of V and are positively invariant because $\nabla V(x)^T f(x) \leq 0$.



Stability follows from this property: choose a level set inside the ball of radius ε, and a ball of radius δ inside this level set. Trajectories starting in $\mathcal{B}_\delta$ can't leave $\mathcal{B}_\varepsilon$ since they remain inside the level set.

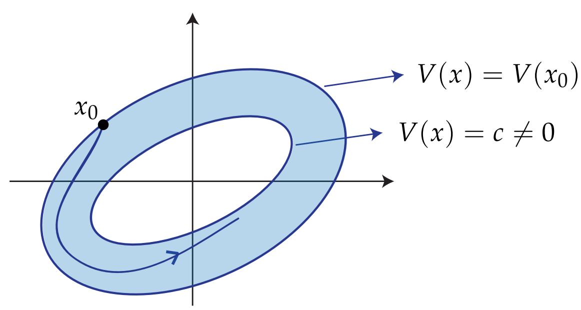


Asymptotic stability:

Since $V(x(t))$ is decreasing and bounded below by 0, we conclude

$$V(x(t)) \to c \geq 0.$$

We will show $c = 0$ (*i.e.*, $x(t) \to 0$) by contradiction. Suppose $c \neq 0$:



Let

$$\gamma \triangleq \min_{\{x:\, c \leq V(x) \leq V(x_0)\}} -\dot{V}(x) \ > 0$$

where the maximum exists because it is evaluated over a bounded[2] set, and is positive because $\dot{V}(x) < 0$ away from $x = 0$. Then,

$$\dot{V}(x) \leq -\gamma \implies V(x(t)) \leq V(x_0) - \gamma t,$$

which implies $V(x(t)) \ < \ 0$ for $t > \frac{V(x_0)}{\gamma}$ – a contradiction because $V \geq 0$. Therefore, $c = 0$ which implies $x(t) \to 0$.

[2] By positive definiteness of V, the level sets $\{x : V(x) \leq \text{constant}\}$ are bounded when the constant is sufficiently small. Since we are proving *local* asymptotic stability we can assume x_0 is close enough to the origin that the constant $V(x_0)$ is sufficiently small.

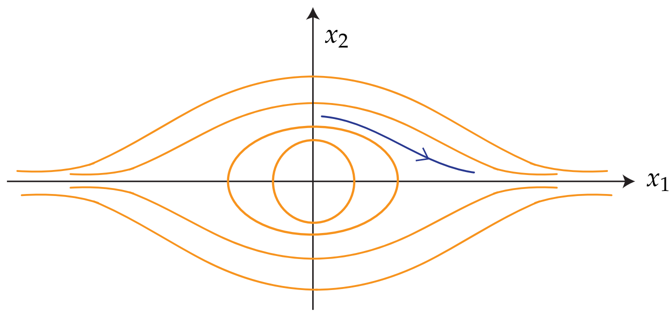
Global asymptotic stability:

Why do we need radial unboundedness?

Example:

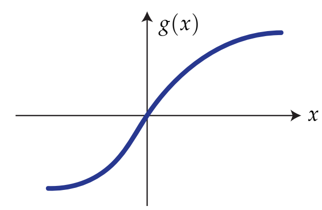
$$V(x) = \frac{x_1^2}{1 + x_1^2} + x_2^2 \tag{2}$$

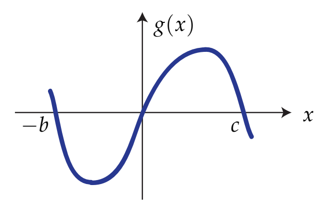
Set $x_2 = 0$, let $x_1 \to \infty$: $V(x) \to 1$ (not radially unbounded). Then Ω_c is not a bounded set for $c \geq 1$:



Therefore, $x_1(t)$ may grow unbounded while $V(x(t))$ is decreasing.

Finding Lyapunov Functions





Example:

$$\dot{x} = -g(x) \quad x \in \mathbb{R}, \; xg(x) > 0 \;\; \forall x \neq 0 \tag{3}$$

$V(x) = \frac{1}{2}x^2$ is positive definite and radially unbounded.

$\dot{V}(x) = -xg(x)$ is negative definite. Therefore $x = 0$ is globally asymptotically stable.

If $xg(x) > 0$ only in $(-b, c) - \{0\}$, then take $D = (-b, c)$

$\implies x = 0$ is locally asymptotically stable.

There are other equilibria where $g(x) = 0$, so we know global asymptotic stability is not possible.

Example:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_2 - g(x_1) \quad a \geq 0, \; xg(x) > 0 \;\; \forall x \in (-b, c) - \{0\} \end{aligned} \tag{4}$$

The pendulum is a special case with $g(x) = \sin(x)$.

The choice $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ doesn't work because $\dot{V}(x)$ is sign indefinite (show this).

The function

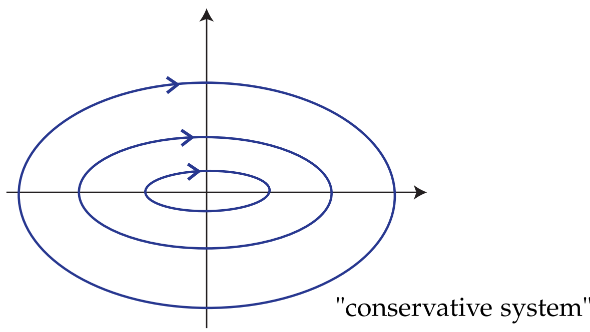
$$V(x) = \int_0^{x_1} g(y)dy + \frac{1}{2}x_2^2$$

is positive definite on $D = (-b, c) - \{0\}$ and

$$\begin{aligned}
\dot{V}(x) &= \frac{\partial V}{\partial x} f(x) \\
&= \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} f(x) \\
&= \begin{bmatrix} g(x_1) & x_2 \end{bmatrix} f(x) \\
&= g(x_1)x_2 - a x_2^2 - x_2 g(x_1) \\
&= -a x_2^2
\end{aligned}$$

is negative semidefinite $\Longrightarrow$ stable.

If $a = 0$, no asymptotic stability because $\dot{V}(x) = 0 \Longrightarrow V(x(t)) = V(x(0))$.



If $a > 0$, (4) is asymptotically stable but the Lyapunov function above doesn't allow us to reach that conclusion. We need either another V with negative definite $\dot{V}$, or the Lasalle-Krasovskii Invariance Principle to be discussed in the next lecture.