ME 6402 – Lecture 7 [1]

MATHEMATICAL BACKGROUND

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Overview:

- Existence and Uniqueness of ODEs
- Lipschitz continuity
- Normed linear spaces
- Fixed point theorems
- Contraction mappings

Additional Reading:

- Sastry, Chapter 3
- Khalil, Chapter 3 and Appendix B

Clarification

A k-dimensional manifold in $\mathbb{R}^n$ $(1 \leq k < n)$ is informally the solution to

$$\eta(x) = 0$$

with $\eta : \mathbb{R}^n \to \mathbb{R}^{n-k}$ sufficiently smooth. Last class, we said that $z = h(y)$ is a *center manifold* for the transformed system $y \in \mathbb{R}^k$ and $z \in \mathbb{R}^{n-k}$, characterized as the solution to $w(x) \triangleq z(x) - h(y(x)) = 0$. Informally, we are constraining $z \in \mathbb{R}^{n-k}$ which allows us to only consider the dynamics of $y \in \mathbb{R}^k$.

Example:
The unit circle:

$$\{x \in \mathbb{R}^2 \text{ s.t. } \eta(x) \triangleq x_1^2 + x_2^2 - 1 = 0\}$$

is a one-dimensional manifold in $\mathbb{R}^2$.

The unit sphere:

$$\{x \in \mathbb{R}^n \text{ s.t. } \eta(x) \triangleq \sum_{i=1}^{n} x_i^2 - 1 = 0\}$$

is a $n-1$ dimensional manifold in $\mathbb{R}^n$.

Mathematical Background

Sastry, Chapter 3

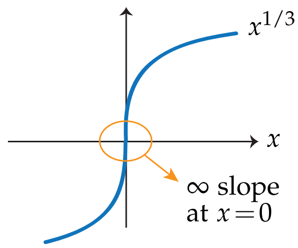
$$\dot{x} = f(x) \quad x(0) = x_0 \tag{1}$$

Do solutions exist? Are they unique?

- If $f(\cdot)$ is continuous (C^0) then a solution exists, but C^0 is not sufficient for uniqueness.

  Example: $\dot{x} = x^{\frac{1}{3}}$ with $x(0) = 0$

$$x(t) \equiv 0,\ x(t) = \left(\frac{2}{3}t\right)^{\frac{3}{2}} \text{ are both solutions}$$



- Sufficient condition for uniqueness: "Lipschitz continuity" (more restrictive than C^0)

$$|f(x) - f(y)| \leq L|x - y| \tag{2}$$

Definition: $f(\cdot)$ is *locally Lipschitz* if every point x^0 has a neighborhood where (2) holds for all x, y in this neighborhood for some L.

Example: $(\cdot)^{\frac{1}{3}}$ is NOT locally Lipschitz (due to ∞ slope)

$(\cdot)^3$ *is* locally Lipschitz:

$$x^3 - y^3 = \underbrace{(x^2 + xy + y^2)}_{\text{in any nbhd of } x^0 \text{, we can find } L \text{ to upper bound this}} (x - y)$$

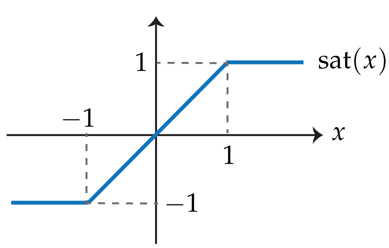
$$\implies |x^3 - y^3| \leq L|x - y|$$

- If $f(\cdot)$ is continuously differentiable (C^1), then it is locally Lipschitz.

Examples: x^3, x^2, e^x, etc.

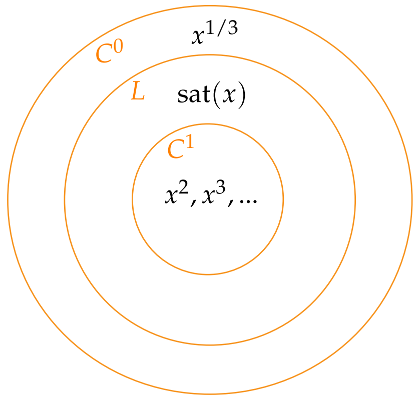
The converse is not true: local Lipschitz $\not\Rightarrow C^1$

Example:



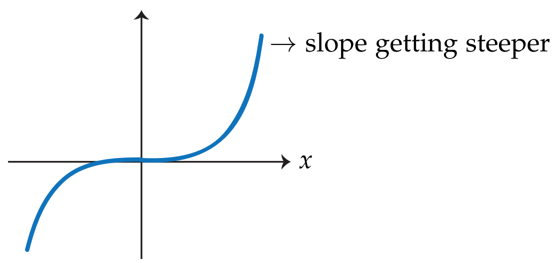
Not differentiable at $x = \mp 1$, but locally Lipschitz:

$$|\operatorname{sat}(x) - \operatorname{sat}(y)| \leq |x - y| \qquad (L = 1).$$



Definition continued: $f(\cdot)$ is *globally Lipschitz* if (2) holds $\forall x, y \in \mathbb{R}^n$ (*i.e.*, the same L works everywhere).

Examples: sat$(\cdot)$ is globally Lipschitz. $(\cdot)^3$ is not globally Lipschitz:



- Suppose $f(\cdot)$ is C^1. Then it is globally Lipschitz iff $\frac{\partial f}{\partial x}$ is bounded.

$$L = \sup_x |f'(x)|$$

Preview of existence theorems:

1. $f(\cdot)$ is $C^0 \implies$ existence of solution $x(t)$ on finite interval $[0, t_f)$.
2. $f(\cdot)$ locally Lipschitz $\implies$ existence and uniqueness on $[0, t_f)$.
3. $f(\cdot)$ globally Lipschitz $\implies$ existence and uniqueness on $[0, \infty)$.

Examples:

- $\dot{x} = x^2$ (locally Lipschitz) admits unique solution on $[0, t_f)$, but $t_f < \infty$ from Lecture 1 (finite escape).
- $\dot{x} = Ax$ globally Lipschitz, therefore no finite escape

$$|Ax - Ay| \leq L|x - y| \quad \text{with} \quad L = \|A\|$$

The rest of the lecture introduces concepts that are used in proving the existence theorems mentioned above.

Normed Linear Spaces

Definition: $\mathbb{X}$ is a normed linear space if there exists a real-valued norm $|\cdot|$ satisfying:

1. $|x| \geq 0 \;\; \forall x \in \mathbb{X}, \;\; |x| = 0$ iff $x = 0$.
2. $|x + y| \leq |x| + |y| \;\; \forall x, y \in \mathbb{X}$ (triangle inequality)
3. $|\alpha x| = |\alpha| \cdot |x| \;\; \forall \alpha \in \mathbb{R}$ and $x \in \mathbb{X}$.

Definition: A sequence $\{x_k\}$ in $\mathbb{X}$ is said to be a Cauchy sequence if

$$|x_k - x_m| \to 0 \text{ as } k, m \to \infty. \tag{3}$$

Every convergent sequence is Cauchy. The converse is not true.

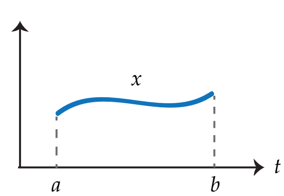
Definition: $\mathbb{X}$ is a Banach space if every Cauchy sequence converges to an element in $\mathbb{X}$.

All Euclidean spaces are Banach spaces.

Example:

$C^n[a, b]$: the set of all continuous functions $[a, b] \to \mathbb{R}^n$ with norm:

$$|x|_C = \max_{t \in [a,b]} |x(t)|$$



1. $|x|_C \geq 0$ and $|x|_C = 0$ iff $x(t) \equiv 0$.

2. $|x+y|_C = \max_{t\in[a,b]} |x(t)+y(t)| \leq \max_{t\in[a,b]} \{|x(t)|+|y(t)|\} \leq |x|_C + |y|_C$

3. $|\alpha \cdot x|_C = \max_{t\in[a,b]} |\alpha| \cdot |x(t)| = |\alpha| \cdot |x|_C$

It can be shown that $C^n[a,b]$ is a Banach space.

Fixed Point Theorems

$$T(x) = x \tag{4}$$

Brouwer's Theorem (Euclidean spaces):

If U is a closed, bounded, convex subset of a Euclidean space and $T : U \to U$ is continuous, then T has a fixed point in U.

Schauder's Theorem (Brouwer's Thm $\to$ Banach spaces):

If U is a closed bounded convex subset of a Banach space $\mathbb{X}$ and $T : U \to U$ is *completely continuous*[2], then T has a fixed point in U.

[2] continuous and for any bounded set $B \subseteq U$ the closure of $T(B)$ is compact

Contraction Mapping Theorem:

If U is a closed subset of a Banach space and $T : U \to U$ is such that

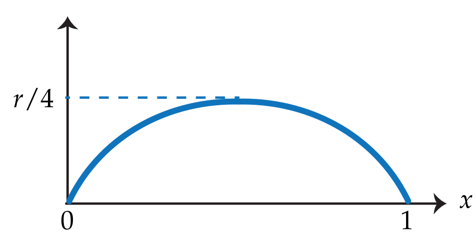
$$|T(x) - T(y)| \leq \rho|x-y| \quad \rho < 1 \quad \forall x, y \in U$$

then T has a unique fixed point in U and the solutions of $x_{n+1} = T(x_n)$ converge to this fixed point from any $x_0 \in U$.

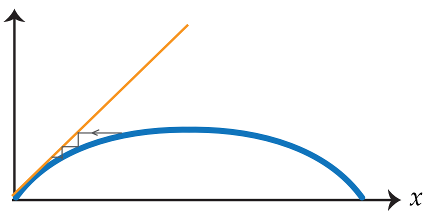
Example: The logistic map (Lecture 5)

$$T(x) = rx(1-x) \tag{5}$$

with $0 \leq r \leq 4$ maps $U = [0,1]$ to U. $|T'(x)| \leq r \;\; \forall x \in [0,1]$, so the contraction property holds with $\rho = r$.



If $r < 1$, the contraction mapping theorem predicts a unique fixed point that attracts all solutions starting in $[0,1]$.



Proof steps for the Contraction Mapping Thm:

1. Show that $\{x_n\}$ formed by $x_{n+1} = T(x_n)$ is a Cauchy sequence. Since we are in a Banach space, this implies a limit x^* exists.

2. Show that $x^* = T(x^*)$.

3. Show that x^* is unique.

Details of each step:

1. $$\begin{aligned}|x_{n+1} - x_n| = |T(x_n) - T(x_{n-1})| &\le \rho|x_n - x_{n-1}| \\ &\le \rho^2|x_{n-1} - x_{n-2}| \\ &\vdots \\ &\le \rho^n|x_1 - x_0|.\end{aligned}$$

$$\begin{aligned}|x_{n+r} - x_n| &\le |x_{n+r} - x_{n+r-1}| + \cdots + |x_{n+1} - x_n| \\ &\le (\rho^{n+r} + \cdots + \rho^n)|x_1 - x_0| \\ &= \rho^n(1 + \cdots + \rho^r)|x_1 - x_0| \\ &\le \rho^n \frac{1}{1-\rho}|x_1 - x_0|\end{aligned}$$

Since $\frac{\rho^n}{1-\rho} \to 0$ as $n \to \infty$, we have $|x_{n+r} - x_n| \to 0$ as $n \to \infty$.

2. $$\begin{aligned}|x^* - T(x^*)| &= |x^* - x_n + T(x_{n-1}) - T(x^*)| \\ &\le |x^* - x_n| + |T(x_{n-1}) - T(x^*)| \\ &\le |x^* - x_n| + \rho|x^* - x_{n-1}|.\end{aligned}$$

Since $\{x_n\}$ converges to x^*, we can make this upper bound arbitrarily small by choosing n sufficiently large. This means that $|x^* - T(x^*)| = 0$, hence $x^* = T(x^*)$.

3. Suppose $y^* = T(y^*)$ $y^* \neq x^*$.

$$|x^* - y^*| = |T(x^*) - T(y^*)| \le \rho|x^* - y^*| \implies x^* = y^*.$$

Thus we have a contradiction.