ME 6402 – Lecture 6 [1]

CENTER MANIFOLD THEORY AND CHAOS IN DISCRETE-TIME

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Overview:

- Center Manifold Theory
- Discrete-time Systems
- Chaos in Discrete-time

Additional Reading:

- Khalil, Chapter 8.1
- Sastry, Chapter 7.6.1

Motivation for Center Manifold Theory

Khalil (Section 8.1), Sastry (Section 7.6.1)

Remark: Center manifold theory is used to study stability of equilibrium points when linearization fails.

Theorem: (4.7 from Khalil). Let $x = 0$ be an equilibrium point for the nonlinear system

$$\dot{x} = f(x)$$

where $f : D \to \mathbb{R}^n$ is continuously differentiable and D is a neighborhood of the origin. Let

$$A = \frac{\partial f}{\partial x}(x) \text{ s.t. }_{x=0}$$

Then,

1. $x^* = 0$ is asymptotically stable if $\Re(\lambda_i) < 0$ for all eigenvalues of A.
2. $x^* = 0$ is unstable if $\Re(\lambda_i) > 0$ for some eigenvalue of A.

Note: If A has some eigenvalues with zero real parts and the rest have negative real parts, then the linearization fails.

Let's assume that A has k eigenvalues with zero real parts and $m = n - k$ eigenvalues with negative real parts:

- One option: analyze a n-th order nonlinear system
- Second option: analyze a lower order nonlinear system (center manifold theory will dictate that this order is the number of eigenvalues such that $\Re(\lambda_i) = 0$)

Mathematical Preliminaries

A k-dimensional manifold in $\mathbb{R}^n$ $(1 \leq k < n)$ is informally the solution to

$$\eta(x) = 0$$

with $\eta : \mathbb{R}^n \to \mathbb{R}^{n-k}$ sufficiently smooth.

Example:
The unit circle:

$$\{x \in \mathbb{R}^2 \text{ s.t. } x_1^2 + x_2^2 = 1\}$$

is a one-dimensional manifold in $\mathbb{R}^2$.

The unit sphere:

$$\{x \in \mathbb{R}^n \text{ s.t. } \sum_{i=1}^{n} x_i^2 = 1\}$$

is a $n-1$ dimensional manifold in $\mathbb{R}^n$.

A manifold is an invariant manifold if:

$$\eta(x(0)) = 0 \implies \eta(x(t)) \equiv 0 \quad \forall t \in [0, t_1) \subset \mathbb{R}$$

where $[0, t_1)$ is any time interval over which $x(t)$ is defined.

Center Manifold Theory

$$\dot{x} = f(x) \quad f(0) = 0 \tag{1}$$

Suppose $A \triangleq \left.\frac{\partial f}{\partial x}\right|_{x=0}$ has k eigenvalues will zero real parts, and $m = n - k$ eigenvalues with negative real parts.

Define $\begin{bmatrix} y \\ z \end{bmatrix} = Tx$ such that

$$TAT^{-1} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

where the eigenvalues of A_1 have zero real parts and the eigenvalues of A_2 have negative real parts.

Rewrite $\dot{x} = f(x)$ in the new coordinates:

$$\begin{aligned} \dot{y} &= A_1 y + g_1(y, z) \\ \dot{z} &= A_2 z + g_2(y, z) \end{aligned} \tag{2}$$

$g_i(0,0) = 0$, $\frac{\partial g_i}{\partial y}(0,0) = 0$, $\frac{\partial g_i}{\partial z}(0,0) = 0$, $i = 1, 2$.

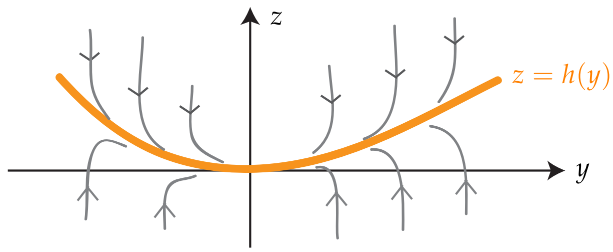
g_1 and g_2 inheret the properties of $\tilde{f}$ in the equation:

$$\dot{x} = f(x) = Ax + \tilde{f}(x)$$

with $\tilde{f}(x) = f(x) - \frac{\partial f}{\partial x}(x)$ s.t. $_{x=0}$, which has the properties $\tilde{f}(0) = 0$ and $\frac{\partial f}{\partial x}(0) = 0$

Theorem 1: There exists an invariant manifold $z = h(y)$ defined in a neighborhood of the origin such that

$$h(0) = 0 \quad \frac{\partial h}{\partial y}(0) = 0.$$



$z = h(y)$ is called a *center manifold* in this case.

Reduced System: $\dot{y} = A_1 y + g_1(y, h(y)) \quad y \in \mathbb{R}^k$

Theorem 2: If $y = 0$ is asymptotically stable (resp., unstable) for the reduced system, then $x = 0$ is asymptotically stable (resp., unstable) for the full system $\dot{x} = f(x)$.

Characterizing the Center Manifold

Define $w \triangleq z - h(y)$ and note that it satisfies

$$\begin{aligned}\dot{w} &= \dot{z} - \frac{\partial h}{\partial y}\dot{y} \\ &= A_2 z + g_2(y, z) - \frac{\partial h}{\partial y}\Big(A_1 y + g_1(y, z)\Big).\end{aligned}$$

The invariance of $z = h(y)$ means that $w = 0$ implies $\dot{w} = 0$. Thus, the expression above must vanish when we substitute $z = h(y)$:

$$\boxed{A_2 h(y) + g_2(y, h(y)) - \frac{\partial h}{\partial y}\Big(A_1 y + g_1(y, h(y))\Big) = 0.}$$

To find $h(y)$ solve this partial differential equation for h as a function on y.

If the exact solution is unavailable, an approximation might be sufficient.

For scalar y, expand $h(y)$ as

$$h(y) = h_2 y^2 + \cdots + h_p y^p + O(y^{p+1})$$

where $h_1 = h_0 = 0$ because $h(0) = \frac{\partial h}{\partial y}(0) = 0$. The notation $O(y^{p+1})$ refers to the higher order terms of power $p + 1$ and above.

Example (8.2 from Khalil):

$$\begin{aligned}\dot{y} &= yz \\ \dot{z} &= -z + ay^2 \quad a \neq 0\end{aligned}$$

This is of the form (2) with $g_1(y,z) = yz$, $g_2(y,z) = ay^2$, $A_2 = -1$. Thus $h(y)$ must satisfy

$$-h(y) + ay^2 - \frac{\partial h}{\partial y} yh(y) = 0.$$

Try $h(y) = h_2 y^2 + O(y^3)$:

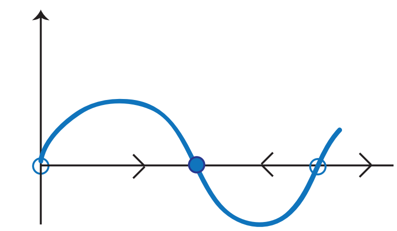
$$\begin{aligned} 0 &= -h_2y^2 + O(y^3) + ay^2 - (2h_2y + O(y^2))y(h_2y^2 + O(y^3)) \\ &= (a - h_2)y^2 + O(y^3) \\ &\Longrightarrow h_2 = a \end{aligned}$$

Reduced System: $\dot{y} = y(ay^2 + O(y^3)) = ay^3 + O(y^4)$.

If $a < 0$, the full systems is asymptotically stable. If $a > 0$ unstable.

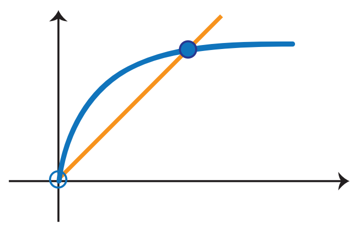
Discrete-Time Models and a Chaos Example

CT: $\dot{x}(t) = f(x(t))$
$f(x^*) = 0$



Asymptotic stability criterion:
$\Re\lambda_i(A) < 0$ where $A \triangleq \frac{\partial f}{\partial x}\Big|_{x=x^*}$
$f'(x^*) < 0$ for first order system

DT: $x_{n+1} = f(x_n)$ $n = 0, 1, 2, \dots$
$f(x^*) = x^*$ ("fixed point")

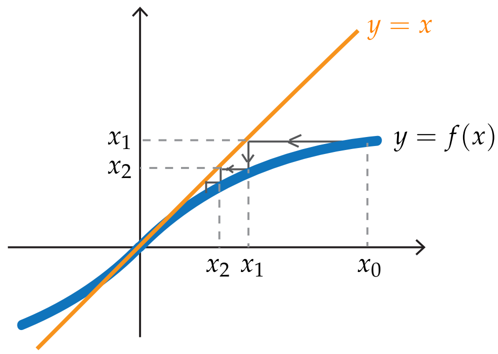


Asymptotic stability criterion:
$|\lambda_i(A)| < 1$ where $A \triangleq \frac{\partial f}{\partial x}\Big|_{x=x^*}$
$|f'(x^*)| < 1$ for first order system

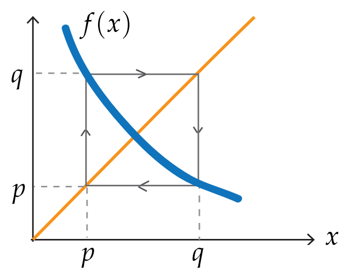
These criteria are inconclusive if the respective inequality is not strict, but for first order systems we can determine stability graphically:

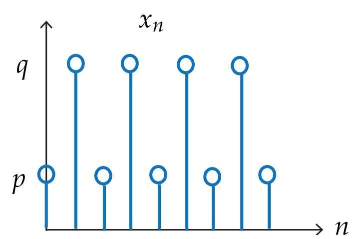
Cobweb Diagrams for First Order Discrete-Time Systems

Example: $x_{n+1} = \sin(x_n)$ has unique fixed point at 0. Stability test above inconclusive since $f'(0) = 1$. However, the "cobweb" diagram below illustrates the convergence of iterations to 0:



In discrete time, even first order systems can exhibit oscillations:





Detecting Cycles Analytically

$$f(p) = q \quad f(q) = p \quad \Longrightarrow \quad f(f(p)) = p \quad f(f(q)) = q$$

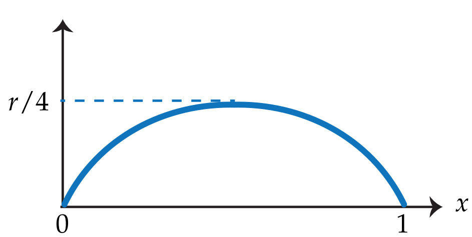
For the existence of a period-2 cycle, the map $f(f(\cdot))$ must have two fixed points in addition to the fixed points of $f(\cdot)$.

Period-3 cycles: fixed points of $f(f(f(\cdot)))$.

Chaos in a Discrete Time Logistic Growth Model

$$x_{n+1} = r(1 - x_n)x_n \tag{3}$$

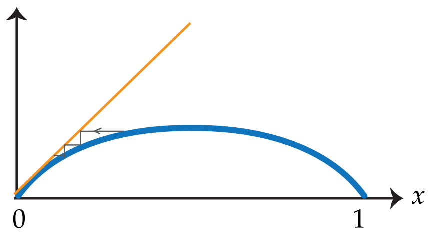
Range of interest: $0 \leq x \leq 1 \quad (x_n > 1 \;\Rightarrow\; x_{n+1} < 0)$



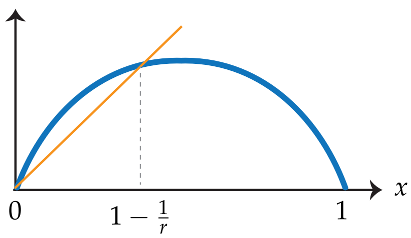
We will study the range $0 \leq r \leq 4$ so that $f(x) = r(1 - x)x$ maps $[0, 1]$ onto itself.

Fixed points: $x = r(1-x)x \;\Rightarrow\; \begin{cases} x^* = 0 \quad \text{and} \\ x^* = 1 - \frac{1}{r} \text{ if } r > 1. \end{cases}$

$\underline{r \leq 1}$: $x^* = 0$ unique and stable fixed point



$\underline{r > 1}$: $x = 0$ unstable because $f'(0) = r > 1$



Note that a transcritical bifurcation occurred at $r = 1$, creating the new equilibrium

$$x^* = 1 - \frac{1}{r}.$$

Evaluate its stability using $f'(x^*) = r(1-2x^*) = 2 - r$.

$$r < 3 \;\Rightarrow\; |f'(x^*)| < 1 \text{ (stable)}$$
$$r > 3 \;\Rightarrow\; |f'(x^*)| > 1 \text{ (unstable)}.$$

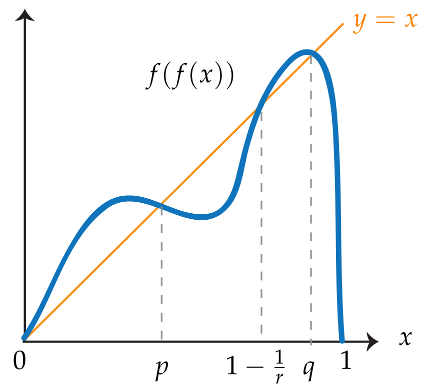
At $r = 3$, a period-2 cycle is born:

$$\begin{aligned} x &= f(f(x)) \\ &= r(1-f(x))f(x) \\ &= r(1-r(1-x)x)r(1-x)x \\ &= r^2x(1-x)(1-r+rx-rx^2) \end{aligned}$$

$$0 = r^2x(1-x)(1-r+rx-rx^2) - x$$

Factor out x and $(x - 1 + \frac{1}{r})$, find the roots of the quotient:

$$p, q = \frac{r + 1 \mp \sqrt{(r-3)(r+1)}}{2r}$$

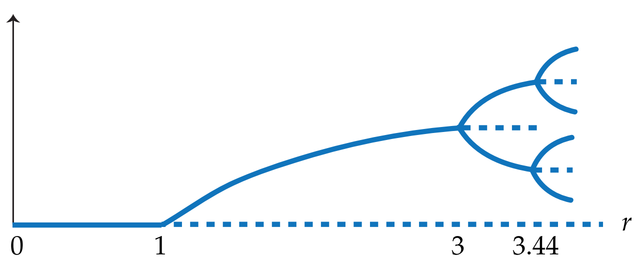


This period-2 cycle is stable when $r < 1 + \sqrt{6} = 3.4494$:

$$\frac{d}{dx} f(f(x))\bigg|_{x=p} = f'(f(p))f'(p) = f'(p)f'(q) = 4 + 2r - r^2$$

$$|4 + 2r - r^2| < 1 \;\Rightarrow\; 3 < r < 1 + \sqrt{6} = 3.4494$$

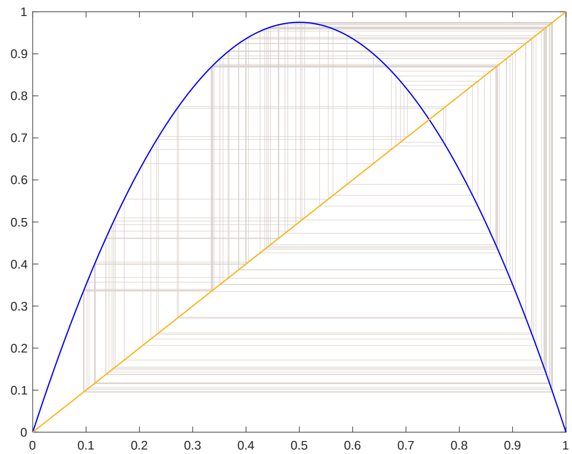
At $r = 3.4494$, a period-4 cycle is born!



$r_1 = 3$	period-2 cycle born
$r_2 = 3.4494$	period-4 cycle born
$r_3 = 3.544$	period-8 cycle born
$r_4 = 3.564$	period-16 cycle born
$\vdots$	
$r_\infty = 3.5699$	

After $r > r_\infty$, chaotic behavior for a window of r, followed by windows of periodic behavior (*e.g.*, period-3 cycle around $r = 3.83$).

Below is the cobweb diagram for $r = 3.9$ which is in the chaotic regime:


0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1