ME 6402 – Lecture 5 [1]

BIFURCATIONS

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[1] Based on notes created by Murat Arcak and licensed under a

Overview:

- Understanding bifurcations in nonlinear systems

Additional Reading:

- Khalil, Chapter 2.7

Motivation

As stated in Khalil, "qualitative behavior of a second-order system is determined by the pattern of its equilibrium points and periodic behavior, as well as by their stability properties". Practically, one must pay special attention to whether or not a system maintains its qualitative behavior under infinitesimally small perturbations. The system is said to be *structurally stable* when stability is maintained. The complement of structural stability is when small perturbations change the equilibrium points or periodic orbits. This phenomenon is known as a *bifurcation*.

Bifurcations

A *bifurcation* is an abrupt change in qualitative behavior as a parameter is varied. Examples: equilibria or limit cycles appearing/disappearing, becoming stable/unstable.

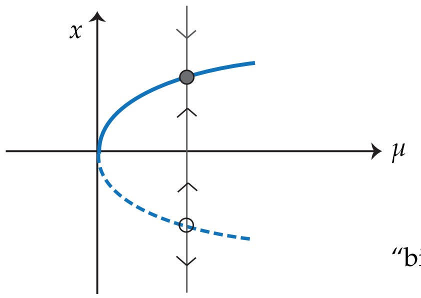
Fold Bifurcation

Also known as "saddle node" or "blue sky" bifurcation.

Example: $\dot{x} = \mu - x^2$

If $\mu > 0$, two equilibria: $x = \mp\sqrt{\mu}$. If $\mu < 0$, no equilibria.

Birfucation diagrams sketch the amplitude of the equilibrium points as a function of the bifurcation parameter. Solid lines represent stable nodes/foci/limit cycles. Dashed lines represent unstable nodes/foci/limit cycles.



"bifurcation diagram"

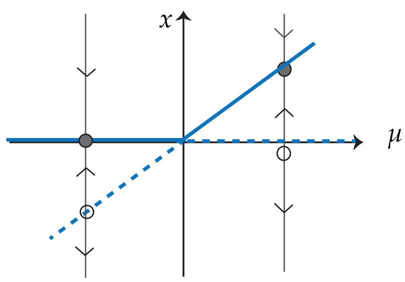
Transcritical Bifurcation

Example: $\dot{x} = \mu x - x^2$

Equilibria: $x = 0$ and $x = \mu$. $\frac{\partial f}{\partial x} = \mu - 2x = \begin{cases} \mu & \text{if } x = 0 \\ -\mu & \text{if } x = \mu \end{cases}$

$\mu < 0$: $x = 0$ is stable, $x = \mu$ is unstable

$\mu > 0$: $x = 0$ is unstable, $x = \mu$ is stable

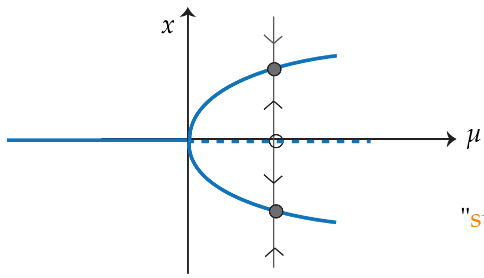


Pitchfork Bifurcation

Example: $\dot{x} = \mu x - x^3$

Equilibria: $x = 0$ for all μ, $x = \mp\sqrt{\mu}$ if $\mu > 0$.

	$\mu < 0$	$\mu > 0$
$\left.\frac{\partial f}{\partial x}\right\|_{x=0} = \mu$	stable	unstable
$\left.\frac{\partial f}{\partial x}\right\|_{x=\mp\sqrt{\mu}} = -2\mu$	N/A	stable

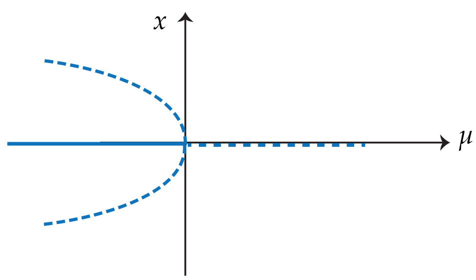


"supercritical pitchfork"

Example: $\dot{x} = \mu x + x^3$

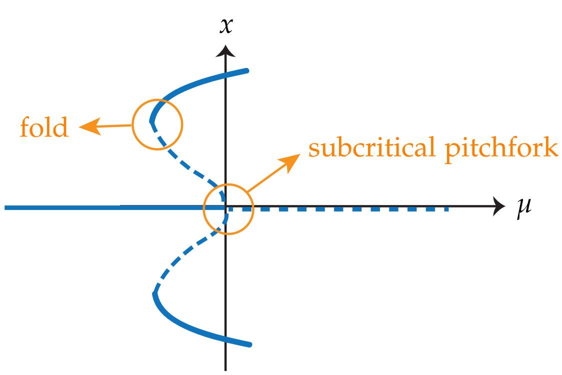
Equilibria: $x = 0$ for all μ, $x = \mp\sqrt{-\mu}$ if $\mu < 0$.

	$\mu < 0$	$\mu > 0$
$\left.\frac{\partial f}{\partial x}\right\|_{x=0} = \mu$	stable	unstable
$\left.\frac{\partial f}{\partial x}\right\|_{x=\mp\sqrt{-\mu}} = -2\mu$	unstable	N/A

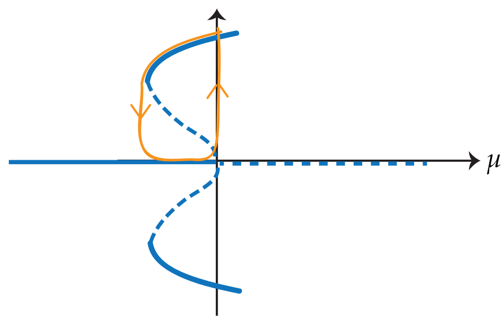


"subcritical pitchfork"

Example: $\dot{x} = \mu x + x^3 - x^5$



Hysteresis arising from a subcritical pitchfork bifurcation:


μ

Example: Bifurcation and hysteresis in perception:

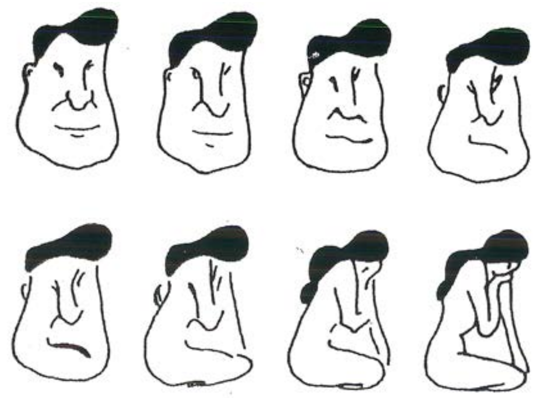


Figure 1: Observe the transition from a man's face to a sitting woman as you trace the figures from left to right, starting with the top row. When does the opposite transition happen as you trace back from the end to the beginning? [Fisher, 1967]

Higher Order Systems

Fold, transcritical, and pitchfork are one-dimensional bifurcations, as evident from the first order examples above. They occur in higher order systems too, but are restricted to a one-dimensional *manifold*.

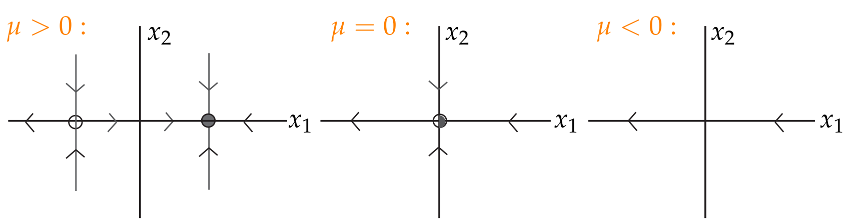
1D subspace: $c_1^T x = \cdots = c_{n-1}^T x = 0$

1D manifold: $g_1(x) = \cdots = g_{n-1}(x) = 0$

Example 1:

$$\begin{aligned}\dot{x}_1 &= \mu - x_1^2 \\ \dot{x}_2 &= -x_2\end{aligned}$$

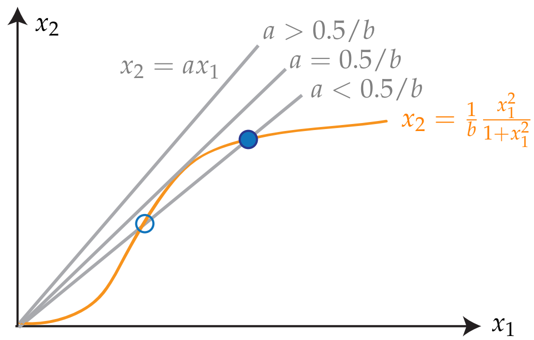
A fold bifurcation occurs on the invariant $x_2 = 0$ subspace:



Example 2: bistable switch (Lecture 1)

$$\dot{x}_1 = -ax_1 + x_2$$
$$\dot{x}_2 = \frac{x_1^2}{1 + x_1^2} - bx_2$$

A fold bifurcation occurs at $\mu \triangleq ab = 0.5$:



Characteristic of one-dimensional bifurcations:

$$\left.\frac{\partial f}{\partial x}\right|_{\mu=\mu^c,\, x=x^*(\mu^c)} \quad \text{has an eigenvalue at zero}$$

where $x^*(\mu)$ is the equilibrium point undergoing bifurcation and μ^c is the critical value at which the bifurcation occurs.

Example 1 above:

$$\left.\frac{\partial f}{\partial x}\right|_{\mu=0,x=0} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow \lambda_{1,2} = \boxed{0}, -1$$

Example 2 above:

$$\left.\frac{\partial f}{\partial x}\right|_{\mu=\frac{1}{2},x_1=1,x_2=a} = \begin{bmatrix} -a & 1 \\ \frac{1}{2} & -b \end{bmatrix} \rightarrow \lambda_{1,2} = \boxed{0}, -(a+b)$$

Hopf Bifurcation

Also called Andronov-Hopf bifurcation or Poincaré-Andronov-Hopf bifurcation to acknowledge the earlier contributions of Poincaré and Andronov.

Two-dimensional bifurcation unlike the one-dimensional types above.

Example: Supercritical Hopf bifurcation

$$\begin{aligned} \dot{x}_1 &= x_1(\mu - x_1^2 - x_2^2) - x_2 \\ \dot{x}_2 &= x_2(\mu - x_1^2 - x_2^2) + x_1 \end{aligned}$$

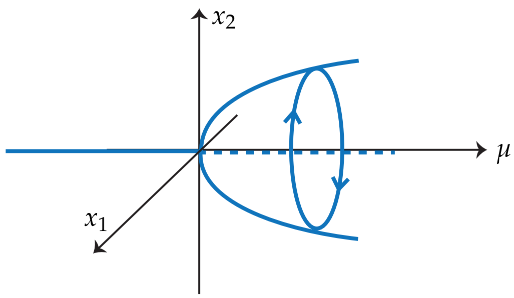
In polar coordinates:

$$\begin{aligned} \dot{r} &= \mu r - r^3 \\ \dot{\theta} &= 1 \end{aligned}$$

Note that a positive equilibrium for the r subsystem means a limit cycle in the (x_1, x_2) plane.

$\mu < 0$: stable equilibrium at $r = 0$

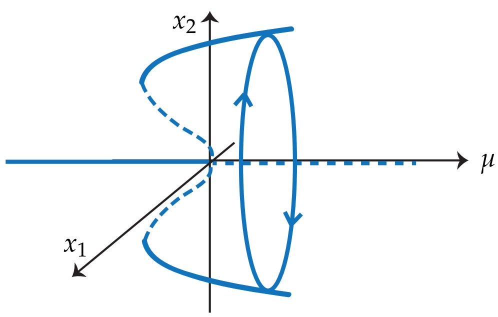
$\mu > 0$: unstable equilibrium at $r = 0$ and stable limit cycle at $r = \sqrt{\mu}$



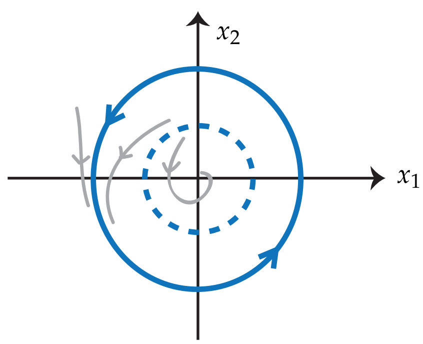
The origin loses stability at $\mu = 0$ and a stable limit cycle emerges.

Example: Subcritical Hopf bifurcation

$$\begin{aligned} \dot{r} &= \mu r + r^3 - r^5 \\ \dot{\theta} &= 1 \end{aligned}$$



Phase portrait for $-0.25 < \mu < 0$:



Characteristic of the Hopf bifurcation:

$$\left.\frac{\partial f}{\partial x}\right|_{\mu=\mu^c,\, x=x^*(\mu^c)} \text{ has complex conjugate eigenvalues on the imaginary axis.}$$