

# ME 6402 – Lecture 4 <sup>1</sup>

## PERIODIC ORBITS IN THE PLANE CONTINUED

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Overview:

- Bendixson's Theorem (Recall)
- Poincaré-Bendixson Theorem
- Index Theory

Additional Reading:

- Khalil, Chapter 2.6

### Remarks

1. The divergence of a continuously differentiable vector field  $f(x)$  for  $x \in \mathbb{R}^n$  is defined as:

$$\operatorname{div}(f(x)) = \nabla \cdot f(x) = \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) \cdot (f_1, \dots, f_n) = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}.$$

Note: the form  $\nabla \cdot f(x)$  is sometimes considered an abuse of notation since we should not apply an operator ( $\nabla$ ) through multiplication.

2. A hyperbolic equilibrium point is a fixed point that does not have any centers. This is equivalent to saying that the linearization has no eigenvalues with zero real part.

### Periodic Orbits in the Plane Continued

Two criteria:

1. Bendixson<sup>2</sup> (absence of periodic orbits)
2. Poincaré-Bendixson (existence of periodic orbits)

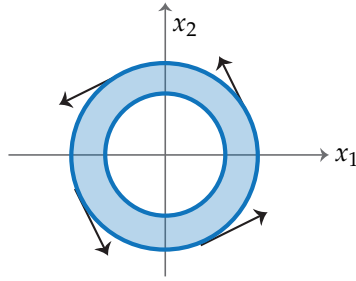
<sup>2</sup> Recall that Bendixson's theorem said that if  $\operatorname{div}(f(x)) \neq 0$  and does not change sign, then  $D$  contains no periodic orbits.

Poincaré-Bendixson Theorem: Suppose  $M$  is compact<sup>3</sup> and positively invariant for the planar, time invariant system  $\dot{x} = f(x)$ ,  $x \in \mathbb{R}^2$ . If  $M$  contains no equilibrium points, then it contains a periodic orbit.

<sup>3</sup> *i.e.*, closed and bounded

Example: Harmonic Oscillator

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1. \end{aligned}$$

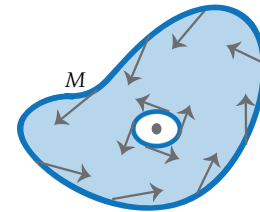


For any  $R > r > 0$ , the ring  $\{x : r^2 \leq x_1^2 + x_2^2 \leq R^2\}$  is compact, invariant and contains no equilibria  $\Rightarrow$  at least one periodic orbit. (We know there are infinitely many in this case.)

The “no equilibrium” condition in the PB theorem can be relaxed as:

“If  $M$  contains one equilibrium which is an unstable focus or unstable node”

*Proof sketch:* Since the equilibrium is an unstable focus or node, we can encircle it with a small closed curve on which  $f(x)$  points outward. Then the set obtained from  $M$  by carving out the interior of the closed curve is positively invariant and contains no equilibrium.



Example 2, Lecture 3: Recall that we were given the system:

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= -2x_1 + x_2 - x_2(x_1^2 + x_2^2)\end{aligned}$$

We were asked to show that  $B_r \triangleq \{x | x_1^2 + x_2^2 \leq r^2\}$  is positively invariant for sufficiently large  $r$ :

$$\begin{aligned}f(x) \cdot n(x) &= \begin{bmatrix} x_1 + x_2 - x_1(x_1^2 + x_2^2) \\ -2x_1 + x_2 - x_2(x_1^2 + x_2^2) \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= x_1^2 + x_1x_2 - x_1^2(x_1^2 + x_2^2) - 2x_1x_2 + x_2^2 - x_2^2(x_1^2 + x_2^2) \\ &= -x_1x_2 + (x_1^2 + x_2^2) - (x_1^2 + x_2^2)^2\end{aligned}$$

which used the inequality

$$|2x_1x_2| \leq x_1^2 + x_2^2,$$

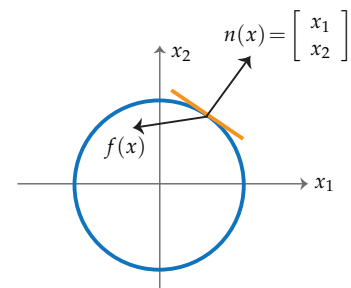
to arrive at the final condition:

$$\begin{aligned}f(x) \cdot n(x) &\leq \frac{1}{2}(x_1^2 + x_2^2) + (x_1^2 + x_2^2) - (x_1^2 + x_2^2)^2 \\ &= \frac{3}{2}r^2 - r^4\end{aligned}$$

Therefore,  $f(x) \cdot n(x) \leq \frac{3}{2}r^2 - r^4 \leq 0$  if  $r^2 \geq \frac{3}{2}$ .

This is a special case of the Cauchy-Schwarz inequality:  $|\langle a, b \rangle| \leq \|a\| \|b\|$  with  $a = (x_1, x_2)$  and  $b = (x_2, x_1)$ :

$$\begin{aligned}|x_1x_2 + x_2x_1| &\leq \sqrt{(x_1^2 + x_2^2)(x_1^2 + x_2^2)} \\ |2x_1x_2| &\leq x_1^2 + x_2^2\end{aligned}$$



So, we can conclude that  $B_r$  is positively invariant for  $r \geq \sqrt{\frac{3}{2}}$  and contains the equilibrium  $x = 0$ .

$$\left. \frac{\partial f}{\partial x} \right|_{x=0} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \quad \lambda_{1,2} = 1 \mp j\sqrt{2} \quad \text{unstable focus.}$$

Therefore,  $B_r$  must contain a periodic orbit.

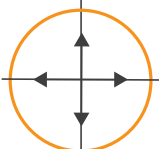
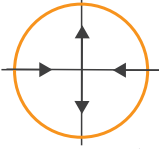
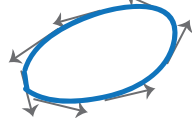
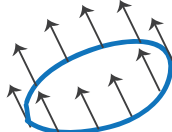
A more general form of the PB Theorem states that, for time invariant, planar systems, bounded trajectories converge to equilibria, periodic orbits, or unions of equilibria connected by trajectories.

*Corollary:* No chaos for time invariant planar systems.

### *Index Theory*

Again, applicable only to planar systems.

Definition (index): The index of a closed curve is  $k$  if, when traversing the curve in one direction,  $f(x)$  rotates by  $2\pi k$  in the same direction. The index of an equilibrium is defined to be the index of a small curve around it that doesn't enclose another equilibrium.

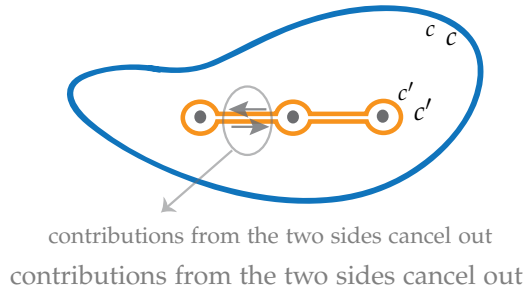
<u>type of equilibrium or curve</u>	<u>index</u>	
node, focus, center	+1	
saddle	-1	
any closed orbit	+1	
a closed curve not encircling any equilibria	0	

The last claim (index = 0) follows from the following observations:

- Continuously deforming a closed curve without crossing equilibria leaves its index unchanged.
- A curve not encircling equilibria can be shrunk to an arbitrarily small one, so  $f(x)$  can be considered constant.

**Theorem:** The index of a closed curve is equal to the sum of indices of the equilibria inside.

*Graphical proof:* Shrinking curve  $c$  to  $c'$  below without crossing equilibria does not change the index. The index of  $c'$  is the sum of the indices of the curves encircling the equilibria because the thin "pipes" connecting these curves do not affect the index of  $c'$ .



The following corollary is useful for ruling out periodic orbits (like Bendixson's Theorem studied in the previous lecture):

*Corollary:* Inside any periodic orbit there must be at least one equilibrium and the indices of the equilibria enclosed must add up to +1.

Example (from last lecture):

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\delta x_2 + x_1 - x_1^3 + x_1^2 x_2 \quad \delta > 0\end{aligned}$$

**Bendixson's Criterion:** No periodic orbit can lie entirely in one of the regions  $x_1 \leq -\sqrt{\delta}$ ,  $-\sqrt{\delta} \leq x_1 \leq \sqrt{\delta}$ , or  $x_1 \geq \sqrt{\delta}$ .

Now apply the corollary above.

Equilibria:  $(0,0)$ ,  $(\mp 1,0)$ . To find their indices evaluate the Jacobian:

$$\left. \frac{\partial f}{\partial x} \right|_{x=(0,0)} = \begin{bmatrix} 0 & 1 \\ 1 & -\delta \end{bmatrix} \quad \lambda^2 + \delta \lambda \underbrace{-1}_{<0} = 0.$$

The eigenvalues are real and have opposite signs, therefore  $(0,0)$  is a saddle: index = -1.

$$\left. \frac{\partial f}{\partial x} \right|_{x=(\mp 1,0)} = \begin{bmatrix} 0 & 1 \\ -2 & 1-\delta \end{bmatrix} \quad \lambda^2 + (\delta-1)\lambda \underbrace{+2}_{>0} = 0.$$

The eigenvalues are either real with the same sign (node) or complex conjugates (focus or center), therefore  $(\mp 1,0)$  each has index = +1.

Thus, the corollary above rules out the periodic orbit in the middle plot below. It does not rule out the others, but does not prove their existence either. Bendixson's Criterion rules out neither of the three.

