ME 6402 – Lecture 3[1]

PHASE POTRAITS OF NONLINEAR SYSTEMS NEAR HYPERBOLIC EQUILIBRIA

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[1] Based on notes created by Murat Arcak and licensed under a

Overview:

- Hartman-Grobman Theorem
- Bendixson's Theorem
- Invariant Sets

Additional Reading:

- Khalil, Chapter 2

Review: Phase Portraits of Linear Systems: $\dot{x} = Ax$

Depending on the eigenvalues of A, there are two main forms for $J = T^{-1}AT$:

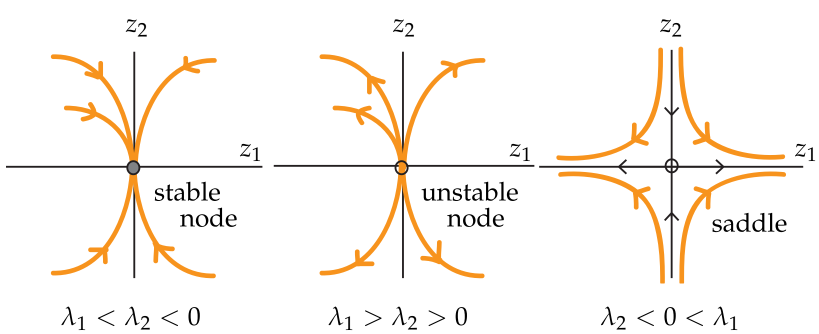
1. Distinct real eigenvalues

$$T^{-1}AT = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

In $z = T^{-1}x$ coordinates:

$$\dot{z}_1 = \lambda_1 z_1, \;\; \dot{z}_2 = \lambda_2 z_2.$$

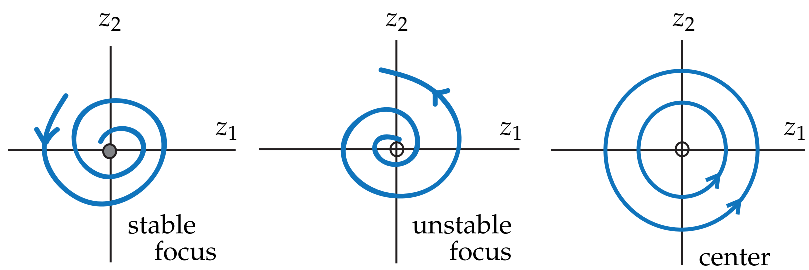
The equilibrium is called a *node* when λ_1 and λ_2 have the same sign (*stable* node when negative and *unstable* when positive). It is called a *saddle point* when λ_1 and λ_2 have opposite signs.



2. Complex eigenvalues: $\lambda_{1,2} = \alpha \mp j\beta$

$$T^{-1}AT = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

$$\begin{aligned} \dot{z}_1 &= \alpha z_1 - \beta z_2 \\ \dot{z}_2 &= \alpha z_2 + \beta z_1 \end{aligned} \quad \rightarrow \quad \text{polar coordinates} \quad \rightarrow \quad \begin{aligned} \dot{r} &= \alpha r \\ \dot{\theta} &= \beta \end{aligned}$$



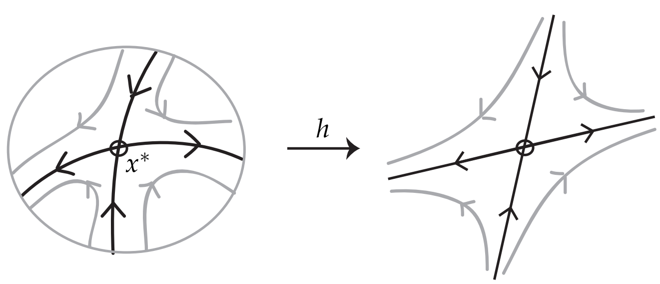
$\alpha < 0$ $\alpha > 0$ $\alpha = 0$

The phase portraits above assume $\beta > 0$ so that the direction of rotation is counter-clockwise: $\dot{\theta} = \beta > 0$.

Phase Portraits of Nonlinear Systems Near Hyperbolic Equilibria

Definition: *Hyperbolic Equilibrium.* Linearization has no eigenvalues on the imaginary axis

Phase portraits of nonlinear systems near hyperbolic equilibria are qualitatively similar to the phase portraits of their linearization. According to the Hartman-Grobman Theorem (below) a "continuous deformation" maps one phase portrait to the other.



Theorem: Hartman-Grobman Theorem.
If x^* is a hyperbolic equilibrium of $\dot{x} = f(x), x \in \mathbb{R}^n$, then there exists a *homeomorphism*[2] $z = h(x)$ defined in a neighborhood of x^* that maps trajectories of $\dot{x} = f(x)$ to those of $\dot{z} = Az$ where $A \triangleq \left.\frac{\partial f}{\partial x}\right|_{x=x^*}$.

[2] a continuous map with a continuous inverse

The hyperbolicity condition can't be removed:

Example:

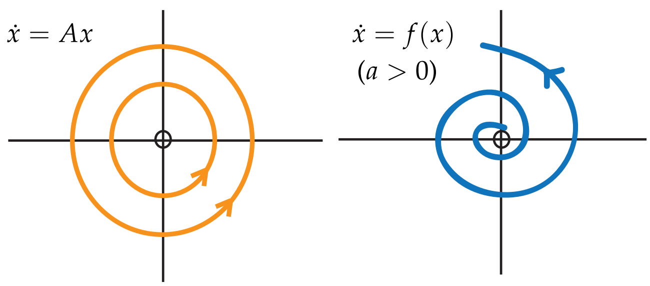
$$\begin{aligned}\dot{x}_1 &= -x_2 + ax_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= x_1 + ax_2(x_1^2 + x_2^2)\end{aligned} \implies \begin{aligned}\dot{r} &= ar^3 \\ \dot{\theta} &= 1\end{aligned}$$

$$x^* = (0,0) \qquad A = \frac{\partial f}{\partial x}\bigg|_{x=x^*} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

This can be equivalently written in vector form as

$$\dot{x} = \begin{bmatrix} -x_2 + ax_1(x_1^2 + x_2^2) \\ x_1 + ax_2(x_1^2 + x_2^2) \end{bmatrix}$$

There is no continuous deformation that maps the phase portrait of the linearization to that of the original nonlinear model:



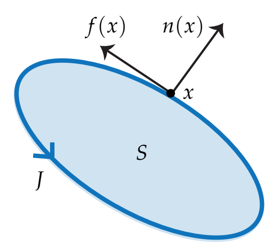
Periodic Orbits in the Plane

Theorem: Bendixson's Theorem. *F*or a time-invariant planar system

$$\dot{x}_1 = f_1(x_1, x_2) \quad \dot{x}_2 = f_2(x_1, x_2),$$

if $\nabla \cdot f(x) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ is not identically zero and does not change sign in a simply connected region D, then there are no periodic orbits lying entirely in D.

Proof: By contradiction. Suppose a periodic orbit J lies in D. Let S denote the region enclosed by J and $n(x)$ the normal vector to J at x. Then $f(x) \cdot n(x) = 0$ for all $x \in J$. By the Divergence Theorem:



$$\underbrace{\int_J f(x) \cdot n(x) d\ell}_{=0} = \underbrace{\iint_S \nabla \cdot f(x) dx.}_{\neq 0}$$

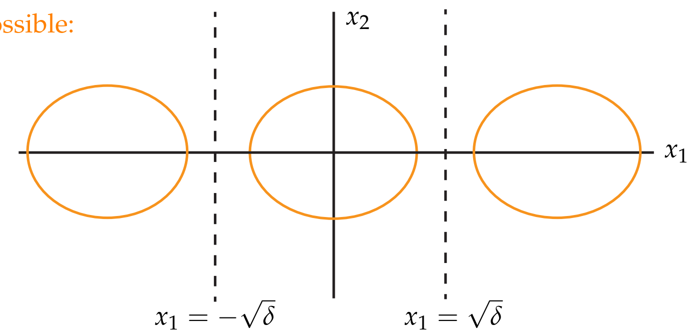
Example:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\delta x_2 + x_1 - x_1^3 + x_1^2 x_2 \quad \delta > 0\end{aligned}$$

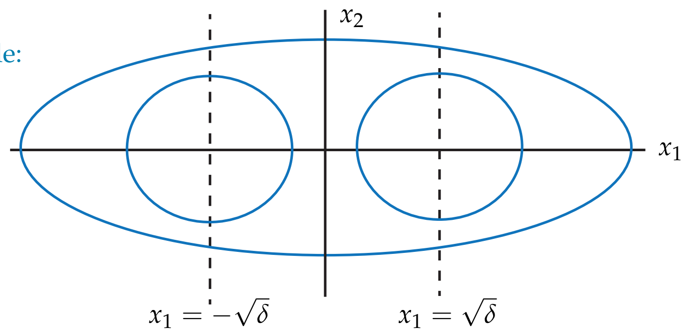
$$\nabla \cdot f(x) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = x_1^2 - \delta$$

Therefore, no periodic orbit can lie entirely in the region $x_1 \leq -\sqrt{\delta}$ where $\nabla \cdot f(x) \geq 0$, or $-\sqrt{\delta} \leq x_1 \leq \sqrt{\delta}$ where $\nabla \cdot f(x) \leq 0$, or $x_1 \geq \sqrt{\delta}$ where $\nabla \cdot f(x) \geq 0$.

not possible:



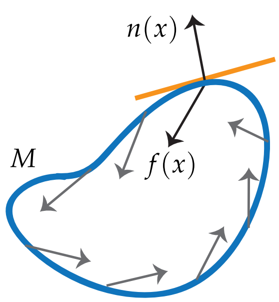
possible:



Invariant Sets

Notation: $\varphi(t, x_0)$ denotes a trajectory of $\dot{x} = f(x)$ with initial condition $x(0) = x_0$.

Definition: A set $M \subset \mathbb{R}^n$ is positively (negatively) invariant if, for each $x_0 \in M$, $\varphi(t, x_0) \in M$ for all $t \geq 0$ ($t \leq 0$).



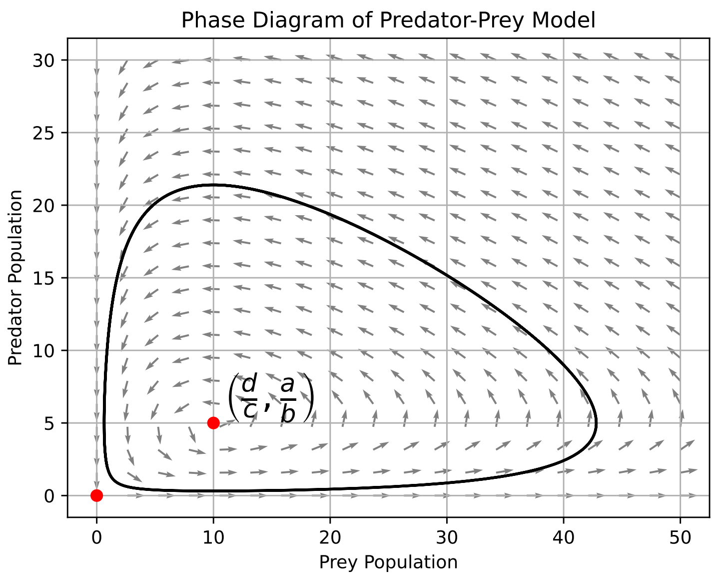
If $f(x) \cdot n(x) \leq 0$ on the boundary then M is positively invariant.

Example 1: A predator-prey model (Lotka-Volterra equations)

$$\begin{aligned} \dot{x} &= (a - by)x && \text{Prey (exponential growth when } y = 0) \\ \dot{y} &= (cx - d)y && \text{Predator (exponential decay when } x = 0) \\ & a, b, c, d, > 0 \end{aligned}$$

The nonnegative quadrant is invariant:

$$\text{(x-axis:)} \quad \begin{bmatrix} ax \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} = 0$$
$$\text{(y-axis:)} \quad \begin{bmatrix} 0 \\ -dy \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix} = 0$$



Example 2: (Similar to Example 2.8 in Khalil)

$$\dot{x}_1 = x_1 + x_2 - x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = -2x_1 + x_2 - x_2(x_1^2 + x_2^2)$$

Show that $B_r \triangleq \{x | x_1^2 + x_2^2 \leq r^2\}$ is positively invariant for sufficiently large r.

$$\begin{aligned} f(x) \cdot n(x) &= \begin{bmatrix} x_1 + x_2 - x_1(x_1^2 + x_2^2) \\ -2x_1 + x_2 - x_2(x_1^2 + x_2^2) \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= x_1^2 + x_1x_2 - x_1^2(x_1^2 + x_2^2) - 2x_1x_2 + x_2^2 - x_2^2(x_1^2 + x_2^2) \\ &= -x_1x_2 + (x_1^2 + x_2^2) - (x_1^2 + x_2^2)^2 \end{aligned}$$

Next, we can use the inequality

$$|2x_1x_2| \leq x_1^2 + x_2^2,$$

This is a special case of the Cauchy-Schwarz inequality: $|\langle a, b\rangle| \leq \|a\|\|b\|$ with $a = (x_1, x_2)$ and $b = (x_2, x_1)$:

$$|x_1x_2 + x_2x_1| \leq \sqrt{(x_1^2 + x_2^2)(x_1^2 + x_2^2)}$$
$$|2x_1x_2| \leq x_1^2 + x_2^2$$

to arrive at the final condition:

$$
\begin{aligned}
f(x) \cdot n(x) &\leq \frac{1}{2}(x_1^2 + x_2^2) + (x_1^2 + x_2^2) - (x_1^2 + x_2^2)^2 \\
&= \frac{3}{2}r^2 - r^4
\end{aligned}
$$

Therefore, $f(x) \cdot n(x) \leq \frac{3}{2}r^2 - r^4 \leq 0$ if $r^2 \geq \frac{3}{2}$.