ME 6402 – Lecture 25

HIGHER-ORDER CONTROL BARRIER FUNCTIONS

April 10 2025

Overview:

- Introduce the notion of *relative degree* for control barrier functions
- Extend CBFs to systems with relative degree > 1

Additional Reading:

- A. Ames, S. Coogan, M. Egerstedt, G. Notomista, K. Sreenath, and P. Tabuada, "Control Barrier Functions: Theory and Applications," IEEE Transactions on Automatic Control, 2019.

Control Barrier Functions

Definition: *CBF (recall).* A function h with $\mathcal{C} = \{x \mid h(x) \geq 0\}$ is a *control barrier function (CBF)* for $\dot{x} = f(x) + g(x)u$ if there exists a locally Lipschitz function $\alpha : \mathbb{R} \to \mathbb{R}$ satisfying $\alpha(0) = 0$ such that

$$\sup_{u \in \mathbb{R}^m} \nabla h(x)^T (f(x) + g(x)u) \geq -\alpha(h(x)) \quad \text{for all } x \in \mathbb{R}^n. \tag{1}$$

We can also write (1) using Lie derivative notation:

$$\sup_{u \in \mathbb{R}^m} L_f h(x) + L_g h(x) u \geq -\alpha(h(x)) \tag{2}$$

Define

$$U(x) = \{u \in \mathbb{R}^m \mid \nabla h(x)^T (f(x) + g(x)u) \geq -\alpha(h(x))\}. \tag{3}$$

Theorem: CBF (recall). *If h is a control barrier function for $\dot{x} = f(x) + g(x)u$, then the following hold:*

1. *$U(x) \neq \emptyset$ for all x;*
2. *Any Lipschitz feedback control $u : \mathbb{R}^n \to \mathbb{R}^m$ satisfying $u(x) \in U(x)$ renders $\mathcal{C}$ invariant;*
3. *A feedback control is given by*

$$u^*(x) = \begin{cases} 0 & \text{if } L_f h(x) + \alpha(h(x)) \geq 0 \\ \frac{-(L_f h(x) + \alpha(h(x))) L_g h(x)^T}{L_g h(x) L_g h(x)^T} & \text{otherwise.} \end{cases} \tag{4}$$

For $u^*(x)$ to be Lipschitz on some domain, we must certify that $L_g h(x) \neq 0$ everywhere on the domain.

Example 1: (Cart-Pole System Revisited)

Recall the model of the cart-pole system:

$$\begin{aligned}\ddot{p} &= \frac{1}{1+\sin^2\theta}\bigg(u + \dot{\theta}^2\sin\theta - g\sin\theta\cos\theta\bigg) \\ \ddot{\theta} &= \frac{1}{1+\sin^2\theta}\bigg(-u\cos\theta - \dot{\theta}^2\cos\theta\sin\theta + 2g\sin\theta\bigg)\end{aligned} \tag{5}$$

Unlike last lecture, suppose we want y to satisfy $-L \leq p \leq L$. Try

$$h(x) = \frac{1}{2}(-p^2 + L^2) \tag{6}$$

$$\alpha(s) = \gamma s, \quad \gamma > 0. \tag{7}$$

But $\nabla h(x)^T g(x) \equiv 0$. Then h cannot be a CBF because the control input vanishes from the CBF condition:

$$\sup_{u\in\mathbb{R}^m} L_f h(x) + L_g h(x) u \geq -\alpha(h(x))$$

For systems such that $\dot{h}(x)$ does *not* depend on u, we need h that depends on more state variables. There is a systematic way to do this. Suppose h satisfies $L_g h(x) \equiv 0$ and cannot be used as a CBF. Define:

$$\Psi_1(x) = L_f h(x) + \alpha_1(h(x))$$

for some Lipschitz α_1 satisfying $\alpha(0) = 0$, and let

$$\mathcal{C}_1 = \{x \mid \Psi_1(x) \geq 0\}$$

This higher-order CBF is then enforced by the condition:

$$\sup_{u\in\mathbb{R}^m} L_f\Psi_1(x) + L_g\Psi_1(x)u \geq -\alpha_2(\Psi_1(x))$$

Lemma: Higher-Order CBF Invariance. *Suppose $u(x)$ is a feedback control law such that $\mathcal{C}_1$ is invariant. Then $\mathcal{C} \bigcap \mathcal{C}_1$ is also invariant, where $\mathcal{C} = \{x \mid h(x) \geq 0\}$.*

Proof. Consider $x_0 \in \mathcal{C} \bigcap \mathcal{C}_1$ and let $x(t)$ be a corresponding closed-loop trajectory. Then $x(t) \in \mathcal{C}_1$ for all $t \geq 0$ by assumption, and therefore

$$\dot{h}(x(t)) = L_f h(x(t)) \geq -\alpha_1(h(x(t))).$$

Since $h(x_0) \geq 0$ by assumption, $h(x(t)) \geq 0$ for all $t \geq 0$ by the Comparison Lemma. □

Question: How can we ensure that $\mathcal{C}_1$ is invariant?
Answer: Use $\Psi_1(x)$ as a CBF!

- If $\nabla\Psi_1(x)^T g(x) \equiv 0$, repeat the process, defining $\Psi_2(x) = \nabla\Psi_1(x)^T f(x) + \alpha_2(\Psi_1(x))$.
- $h(x)$ is called a *high-order CBF of degree* r when this process ends with a CBF $\Psi_{r-1}(x)$.

How many times will we repeat, i.e., what is r? This is related to relative degree.

- *Least relative degree* r is the minimum relative degree over all states x. Therefore $L_g\Psi_{r-1}(x) = L_g L_f^{r-2} h(x) \neq 0$ for some x, but not necessarily all x.

For the previous construction to lead to a valid CBF, we need:

$$L_f\Psi_{r-1}(x) + \alpha_r(\Psi_r(x)) \geq 0, \quad \text{whenever } L_g\Psi_{r-1}(x) = 0$$

- States where $L_g\Psi_{r-1}(x) = 0$ become important to pay attention to (more on this later)

Example 2: Consider the double integrator $\ddot{x}_1 = u$, i.e., $\dot{x}_1 = x_2$, $\dot{x}_2 = u$. Explicitly, this is written in control-affine form:

$$\dot{x} = \begin{bmatrix} x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Suppose that we want $x_1 \leq L$ always. Choose:

$$h(x) = L - x_1$$

We can check the relative degree of the control barrier function:

$$\nabla h(x)^T g(x) = \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \equiv 0$$

This is the same thing as differentiating $h(x)$ until u appears:

$$\dot{h}(x) = -\dot{x}_1 = -x_2$$
$$\ddot{h}(x) = -\dot{x}_2 = -u$$

Thus, since the relative degree is > 1, we will need a higher order CBF.

Choosing $\alpha_1(s) = \gamma_1 s$, the higher-order CBF Ψ_1 is defined as:

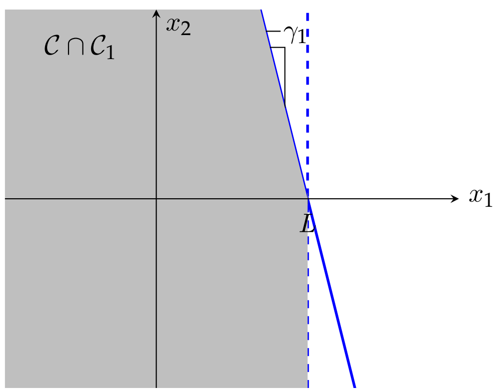
$$\begin{aligned}\Psi_1(x) &= \underbrace{L_f h(x) + \cancel{L_g h(x)}}_{\dot{h}} + \alpha_1(h(x)) \\ &= \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ u \end{bmatrix} + \gamma(L - x_1) \\ &= -x_2 + \gamma(L - x_1)\end{aligned}$$

We can verify that Ψ_1 is a CBF by checking the condition:,

$$L_g \Psi_1 = \nabla \Psi_1(x)^T g(x) = -1$$

Thus we can use $\Psi_1(x)$ as a valid CBF. Explicitly, our safe sets are defined as:

$$\begin{aligned}\mathcal{C} &= \{x \mid h(x) \geq 0\} = \{x \mid x_1 \leq L\} \\ \mathcal{C}_1 &= \{x \mid \Psi_1(x) \geq 0\} = \{x \mid -x_2 + \gamma(L - x_1) \geq 0\}\end{aligned}$$



Explicitly, the higher-order CBF Φ_1 can be inforced via the condition (and taking $\alpha_2(s) = \gamma_2 s$):

$$\begin{aligned}\dot{\Psi}_1 &\geq -\alpha_2(\Psi_1(x)) \\ -\dot{x}_2 - \gamma_1 \dot{x}_1 &\geq -\gamma_2(-x_2 + \gamma_1(L - x_1)) \\ -u - \gamma_1 x_2 &\geq -\gamma_2(-x_2 + \gamma_1(L - x_1))\end{aligned}$$

Example 3: Let's try $\ddot{x}_1 = u$ again, but with the safe set $-L \geq x \leq L$. Choose:

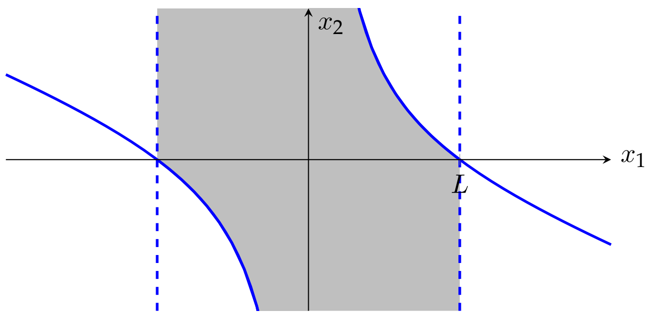
$$h(x) = \frac{1}{2}(-x_1^2 + L^2).$$

Then,

$$\begin{aligned}\dot{h}(x) &= -x_1 \dot{x}_1 = -x_1 x_2 \\ \ddot{h}(x) &= -\dot{x}_1 x_2 - x_1 \dot{x}_2 \\ &= -x_2^2 - x_1 u\end{aligned}$$

Thus the least relative degree is $r = 2$. This yields the higher-order CBF:

$$\Psi_1(x) = \dot{h} + \alpha_1(h(x)) = -x_1 x_2 + \alpha_1\left(\frac{1}{2}(-x_1^2 + L^2)\right)$$



However, in this example, its important to note that there are states where $L_g\Psi_1(x) = 0$:

$$L_g\Psi_1(x) = \nabla\Psi_1(x)^T g(x) = \begin{bmatrix} -x_2 - \gamma_1 & -x_1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -x_1$$

We can also observe this by taking the derivative of Ψ_1:

$$\begin{aligned} \Psi(x) &= \dot{h} + \alpha_1(h(x)) \\ \dot{\Psi}(x) &= \ddot{h} + \alpha_1'(h(x))\dot{h}(x) \\ &= -x_2^2 \underbrace{-x_1}_{L_g\Psi_1} u + \alpha_1'\left(\frac{1}{2}(-x_1^2 + L^2)\right)(-x_1 x_2) \end{aligned}$$

- It is possible that $L_g L_f h(x) = 0$? Yes! Whenever $x_1 = 0$.
- Is this a problem? We need to investigate further...

We need to see if we can find α_2 such that

$$\dot{\Psi}(x) + \alpha_2(\Psi(x)) \geq 0$$

whenever $x_1 = 0$. Evaluating at $x_1 = 0$:

$$\dot{\Psi}(x) + \alpha_2(\Psi(x))\,|_{x_1=0} = -x_2^2 + \alpha_2(\alpha_1(L^2/2))$$

Thus, it is always possible to find x_2 large enough so that $-x_2^2 + \alpha_2(\alpha_1(L^2/2)) < 0$, regardless of α_1 and α_2, so Ψ is not a valid CBF. What should we do? We have two options:

Option 1: Nothing, except make sure α_1 and α_2 have sufficient slope so that this is only a problem when x_2 is very large. This is practical, but loses theoretical guarantees

Option 2: Try a different higher-order CBF h (next example)

Example 4: Let's consider the same system $\ddot{x}_1 = u$, but we will try the control barrier function:

$$h(x) = \frac{1}{4}(-x_1^4 + L^4), \quad \dot{h}(x) = -x_1^3 x_2, \quad \ddot{h}(x) = 3x_1^2 x_2^2 - x_1^3 u.$$

Let

$$\Psi(x) = \dot{h} + \gamma_1 h = -x_1^3 x_2 + \frac{\gamma_1}{4}(-x_1^4 + L^4)$$
$$\dot{\Psi}(x) = \ddot{h} + \gamma_1 \dot{h} = 3x_1^2 x_2^2 - x_1^3 u + \frac{\gamma_1}{4}(-x_1^3 x_2)$$

Then, still $x_1^3 = 0$ whenever $x_1 = 0$. But,

$$L_f \Psi(x) = 3x_1^2 x_2^2 - \alpha_1 x_1^3 x_2$$

and therefore $L_f \Psi(x) = 0$ whenever $L_g \Psi(x) = 0$. This means that Ψ satisfies the CBF constraint $\sup_u L_f \Psi(x) + L_g \Psi(x) u \geq -\alpha_2(\Psi(x))$ for any α_2, and Ψ is a valid CBF.

Note: The important takeaway is to make sure that $L_f \Psi = 0$ whenever $L_g \Psi = 0$.

Example 5: You will implement a higher-order CBF for the cart-pole system for your homework! :)