ME 6402 – Lecture 23

BARRIER FUNCTIONS

April 3 2025

Overview:

- Introduce the Comparison Lemma
- Define Barrier Functions

Additional Reading:

- A. Ames, S. Coogan, M. Egerstedt, G. Notomista, K. Sreenath, and P. Tabuada, "Control Barrier Functions: Theory and Applications," IEEE Transactions on Automatic Control, 2019.

Motivation

So far, we've discussed how to certify and enforce stability, including:

- Hartman-Grobman Theorem
- Center Manifold Theorem
- Lyapunov Analysis
- Feedback Linearization (input-output linearization or potentially full-state feedback linearization)
- Control Lyapunov Functions

Stability can be thought of as certifying/enforcing that the system will *eventually* converge to a desired state. But what about safety? In comparison, safety can be thought of as certifying/enforcing that the system will *never* enter a dangerous/unsafe state.

We will introduce a new class of functions called **barrier functions** that can be used to certify and enforce safety. The key difference between Lyapunov functions and barrier functions can be summarized as:

$$\underbrace{\dot{V} \leq -\alpha(V(x))}_{\text{Stability}} \qquad \text{versus} \qquad \underbrace{\dot{h} \geq -\alpha(h(x))}_{\text{Safety}}$$

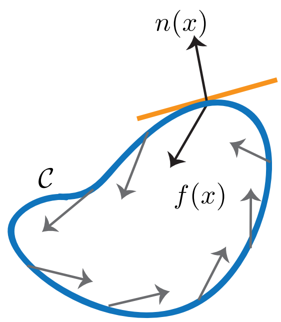
Barrier Functions

The goal of barrier functions is to render a set $\mathcal{C}$ positively invariant[1], which allows us to conclude safety.

[1] **Definition**: *Positively Invariant.* A set $\mathcal{C}$ is *forward (positive) invariant* if for every $x_0 \in \mathcal{C}$, $x(t) \in \mathcal{C}$ for $x(0) = x_0$ and all $t \geq 0$.

Definition: *Safe.* A system $\dot{x} = f(x)$ is *safe* with respect to the set $\mathcal{C}$ if the set $\mathcal{C}$ is forward invariant.

For $\dot{x} = f(x)$, recall from Lecture 5 that we can check positive invariance of a set $\mathcal{C}$ by checking that $n(x)^T f(x) \leq 0$ for all x on the boundary of $\mathcal{C}$ where $n(x)$ is an outward pointing normal vector to the set $\mathcal{C}$.



If $\mathcal{C} = \{x \text{ s.t. } h(x) \geq 0\}$ for some continuously differentiable function h, then $n(x) = -\nabla h(x)$ whenever $\nabla h(x) \neq 0$, and then the previous condition becomes:

$$\nabla h(x)^T f(x) \geq 0 \text{ for all } x \text{ such that } h(x) = 0. \tag{1}$$

However, there are (at least) two potential problems with this approach:

- What if we have a function for which $\nabla h(x) = 0$ for some x on the boundary of $\mathcal{C}$?
- Above condition is only at the boundary and is not good for creating controllers (everything is fine until suddenly it's not)

Intuitive idea of barriers: make sure the system "slows down" as it approaches the boundary of $\mathcal{C}$.

- This lecture: barrier functions for autonomous systems
- Next lecture: control barrier functions for control-affine systems

Definition: *Barrier Function.* A function h with $\mathcal{C} = \{x \text{ s.t. } h(x) \geq 0\}$ is a *barrier function* for $\dot{x} = f(x)$ if there exists a locally Lipschitz function[2] $\alpha : \mathbb{R} \to \mathbb{R}$ satisfying $\alpha(0) = 0$ such that

$$\nabla h(x)^T f(x) \geq -\alpha(h(x)) \quad \text{for } \textit{all } x \in \mathbb{R}^n. \tag{2}$$

[2] When α is also increasing, it is sometimes called an *extended class $\mathcal{K}$ function*; recall our definition of class $\mathcal{K}$ functions from Lecture 12

Using Lie derivative notation, recall $\nabla h(x)^T f(x) = L_f h(x) = \dot{h}(x)$. Thus, this conditions is sometimes written as:

$$\dot{h} \geq -\alpha(h(x)) \quad \text{for } \textit{all } x \in \mathbb{R}^n. \tag{3}$$

- In general, we think of α as being an increasing function, but this is not needed for the theory on the next slide.
- Discussion of "local Lipschitz" requirement at end of lecture.

Intuition and typical use case: Consider a mobile robot, if we want to keep the robot in some region $\mathcal{C}$, we can construct a barrier function $h(x)$ such that $h(x) \geq 0$ everywhere inside of $\mathcal{C}$ and $h(x) = 0$ at the boundaries. Then, we can enforce $L_f h + L_g h u \geq -ch$ with $c > 0$ being some constant. As a rule of thumb, if $c = 0$, then the system must never move towards the boundary. If instead, c is very large, then the system can move rapidly towards the boundary, but the velocity must be still be zero at the boundary, and based on the Lipschitz condition, the system will slow down as it appraoches the boundary.

Theorem: Barrier Function. *If h is a barrier function, then $\mathcal{C} = \{x : h(x) \geq 0\}$ is positively invariant*[3].

[3] and thus the system is *safe* with respect to $\mathcal{C}$

The proof relies on the Comparison Lemma[4] (introduced in Lecture 11). We will re-summarize this Lemma below.

[4] Details on the Comparison Lemma can be found in Khalil, Section 4.4.

Lemma: Comparison Lemma. *Consider the scalar system*

$$\dot{z} = g(z), \quad z(0) = z_0 \tag{4}$$

with locally Lipschitz g. Let $v(t)$ be some continuously differentiable function satisfying

$$\dot{v}(t) \geq g(v(t)) \text{ for all } t \geq 0, \text{ and} \tag{5}$$

$$v(0) \geq z_0. \tag{6}$$

Then $v(t) \geq z(t)$ for all t.

Proof. (Sketch proof of barrier theorem)

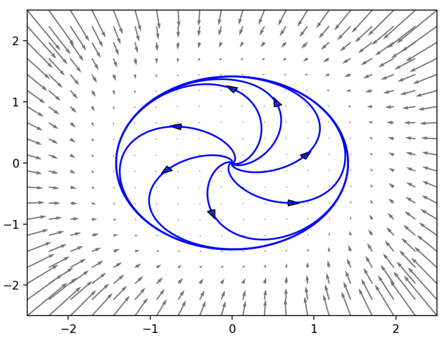
1. Let $x(t)$ be any system trajectory such that $x(0) \in \mathcal{C}$ and define $v(t) = h(x(t))$. Then $\dot{v}(t) = \nabla h(x(t))^T f(x(t)) \geq -\alpha(h(x(t)) = -\alpha(v(t))$, i.e.,
$$\dot{v}(t) \geq -\alpha(v(t)).$$
2. Note that $z(t) \equiv 0$ is a trajectory of $\dot{z} = -\alpha(z)$ since the initial condition $z(0) = 0$ is an equilibrium. Since $v(t) \geq z(0)$, by the Comparison Lemma, $v(t) \geq z(t) = 0$ for all $t \geq 0$, which means $x(t) \in \mathcal{C}$ for all $t \geq 0$.

Example 1: Consider

$$\dot{x}_1 = (a - (x_1^2 + x_2^2))x_1 - x_2 \tag{7}$$
$$\dot{x}_2 = (a - (x_1^2 + x_2^2))x_2 + x_1. \tag{8}$$

In polar coordinates,

$$\dot{r} = r(a - r^2), \quad \dot{\theta} = 1. \tag{9}$$



Let $\mathcal{C} = \{x : h(x) \geq 0\}$ with $h(x) = a - (x_1^2 + x_2^2)$. Then

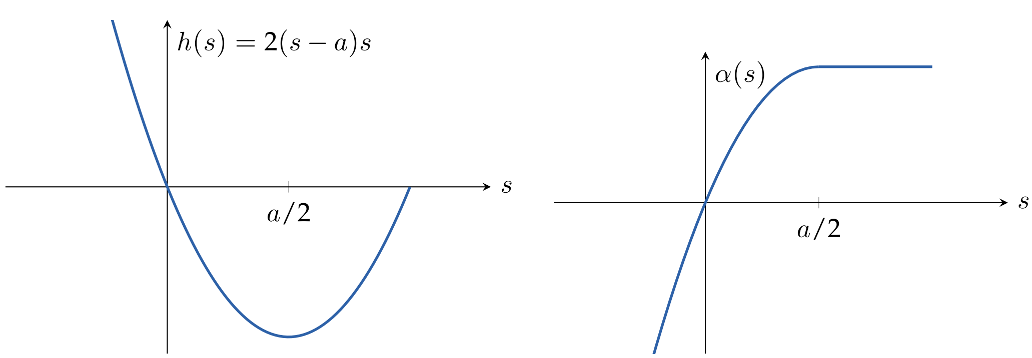
$$\dot{h}(x) = \nabla h(x)^T f(x) = 2(h(x) - a)h(x). \tag{10}$$

For h to be a barrier function, we need to find a Lipschitz continuous function α such that

$$\dot{h}(x) \geq -\alpha(h(x))$$
$$\Updownarrow$$
$$2(s-a)s \geq -\alpha(s)$$

Take

$$\alpha(s) = \begin{cases} -2(s-a)s & \text{if } s \leq a/2 \\ a^2/2 & \text{if } s > a/2 \end{cases} \tag{11}$$



Example 2: Suppose $V(x)$ is a Lyapunov function for the system $\dot{x} = f(x)$. Take $h(x) = C - V(x)$ for some C. Then

$$\mathcal{C} = \{x \text{ s.t. } h(x) \geq 0\} = \{x \text{ s.t. } V(x) \leq C\} \tag{12}$$

We take $\alpha(s) = 0$ and establish positive invariance for $\mathcal{C}$, a sublevel set of V. This choice of α means that trajectories never move closer to the boundary of $\mathcal{C}$, as expected from Lyapunov theory.

Example 3: Consider

$$\dot{x}_1 = (-a + bx_2^2)x_1 \tag{13}$$
$$\dot{x}_2 = (cx_1^2 - d)x_2. \tag{14}$$

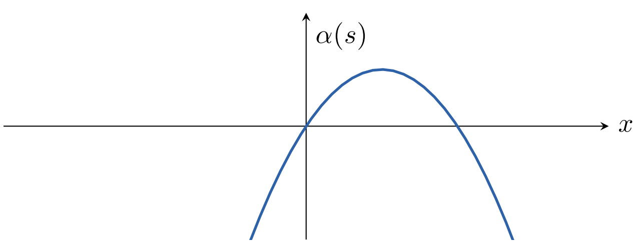
We want to show that the union of the 1st and 3rd quadrants is invariant, *i.e.*, $C = \{(x_1, x_2) \text{ s.t. } h(x) \geq 0\}$ with $h(x) = x_1x_2$. We have

$$\dot{h}(x) = \nabla h(x)^T f(x) = \dot{x}_1 x_2 + x_1 \dot{x}_2 \tag{15}$$
$$= x_1x_2(-a + bx_2^2) + x_1x_2(cx_1^2 - d). \tag{16}$$

Note that, since $(\sqrt{b}x_2 - \sqrt{c}x_1)^2 \geq 0$, then $bx_2^2 + cx_1^2 \geq 2\sqrt{bc}x_1x_2$ and therefore

$$\nabla h(x)^T f(x) \geq (-a - d + 2\sqrt{bc}h(x))h(x). \tag{17}$$

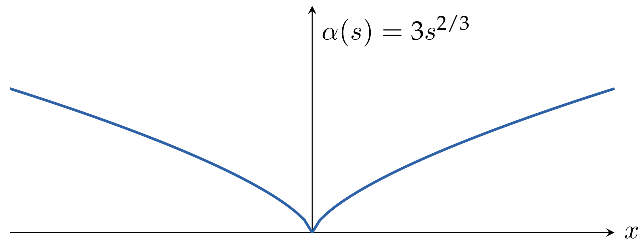
Take $\alpha(s) = -(-a - d + 2s\sqrt{bc})s$.



- Note that $\alpha(0) = 0$, as required. α is not increasing, but this is not an issue.

Note: Local Lipschitzness of α is required for the Comparison Lemma to apply:

Example: Take $\dot{x} = -1$, $h(x) = x^3$ so $C = \{x : h(x) \geq 0\}$. Then $\dot{h}(x) = -h'(x) = -3x^2 = -3h(x)^{2/3}$. It is tempting to take $\alpha(s) = 3s^{2/3}$, a well-defined function satisfying $\alpha(0) = 0$, and it is even increasing for $s \geq 0$. But it is not Lipschitz, and the comparison lemma does not apply.



Further discussion of Lipschitzness[5]:

- It is possible to weaken Lipschitz condition: The key is to ensure that, even if the comparison system $\dot{z} = -\alpha(z)$ with $z(0) = z_0$ has multiple solutions, all solutions remain nonnegative.
- An alternative assumption is to require that $\nabla h(x) \neq 0$ whenever $h(x) = 0$ so that $\nabla h(x)$ always provides a valid normal vector and our original technique (sometimes called Nagumo's theorem) applies. Then, the Comparison Lemma is not required.
- For this alternative, the proof of invariance does not require any other properties of α besides $\alpha(0) = 0$.
- Barriers are a hot topic, but beware that many papers fail to explicitly make either assumption.

[5] See R. Konda, A. Ames, S. Coogan, "Characterizing safety: minimal barrier functions from scalar comparison systems," IEEE Control Systems Letters, 2020, for more details