ME 6402 – Lecture 21

A PRIMER ON CONVEX OPTIMIZATION

March 27 2025

Overview:

- Define optimization problems
- Define convex functions and sets
- Define convex optimization problems

Additional Reading:

- S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.

Motivation

Our motivation for optimization-based control is the formulation:

$$u^* := k(x) = \text{argmin}_u \ \|u\|^2$$
$$\text{subject to} \quad \frac{\partial V}{\partial x}(f(x) + g(x)u) \leq -\varepsilon V(x)$$

This formulation is sometimes called a *control Lyapunov function quadratic program* (CLF-QP). Explicitly, the goal is to find the smallest control input u that stabilizes our system and additionally ensures that the Lyapunov function $V(x)$ decreases at a rate of at least $\varepsilon V(x)$.

To understand how this problem is solved, and when/how we can add additional constraints (such as torque limits) to this problem, we will first dive deeper into optimization problems and convex functions and sets.

Optimization Problems

We often encounter problems of the form

$$\begin{aligned} \text{minimize}_x \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0, \qquad i = 1, \ldots, m \end{aligned} \tag{1}$$

If we are instead interested in maximizing a function $\tilde{f}_0(x)$, we simply define $f_0(x) = -\tilde{f}_0(x)$ to change to a minimization problem.

Equality constraint $f(x) = 0$ is allowed by including two constraints $f(x) \leq 0$ and $-f(x) \leq 0$

where:

- $x \in \mathbb{R}^n$ is an optimization variable,
- f_0 is the objective function, and
- $f_i(x)$ are constraint functions.

The *optimal value* of $f_0(x)$ is the (limit of the) smallest value obtained by $f_0(x)$ on the *feasible set*. A point that achieves the optimal value (*i.e.*,argmin) is an *optimal point*.

Example

Minimum effort stabilization from CLF:
As we mentioned, given the system $\dot{x} = f(x) + g(x)u$ and CLF $V(x)$, we can use the optimization-based controller

$$\begin{aligned} k(x) = &\text{argmin}_u \quad \|u\|^2 \\ &\text{subject to} \quad \frac{\partial V}{\partial x}(f(x) + g(x)u) \leq -\varepsilon A(x), \end{aligned} \tag{2}$$

- ε is user chosen
- $A(x)$ is some positive definition function. $A(x) = x^T x$ or $A(x) = V(x)$ are common choices
- Generally **cannot** consider a strict inequality constraint like $\dot{V}(x) < 0$, hence the need for $\varepsilon A(x)$

Example 2

Finding polynomial Lyapunov functions:
Given system $\dot{x} = f(x)$, solve

$$\begin{aligned} c^* = &\text{argmin}_c \quad 0 \\ &\text{subject to} \quad V(x) \geq \varepsilon_1 A(x) \ \forall x \\ &\qquad\qquad\quad \frac{\partial V}{\partial x} f(x) \leq -\varepsilon_2 A(x) \ \forall x \end{aligned} \tag{3}$$

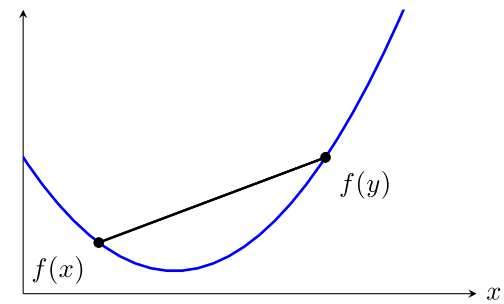
where, *e.g.*, $x \in \mathbb{R}^2$, $V(x) = c_1 x_1^4 + c_2 x_1^3 x_2 + c_3 x_1^2 x_2^2 + c_4 x_1 x_2^3 + c_5 x_2^4 + c_5 x_1^3 + \ldots + c_{n-2} x_1 + c_{n-1} x_2 + c_n$

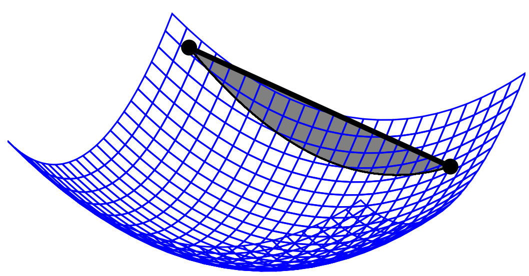
- No cost $\implies$ feasibility question
- "$\forall x$" $\implies$ infinite, uncountable number of constraints

Convex functions and sets

A *convex function* $f : \mathbb{R}^n \to \mathbb{R}$ satisfies for all x, y and all $0 \le \theta \le 1$:

$$f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta)f(y). \tag{4}$$





Example

Can we prove that linear functions are convex? Consider $f(x) = c^T x$ for fixed $c \in \mathbb{R}^n$:

$$\begin{aligned} f(\theta x + (1-\theta)y) &= c^T(\theta x + (1-\theta)y) && (5)\\ &= \theta c^T x + (1-\theta)c^T y && (6)\\ &= \theta f(x) + (1-\theta)f(y), && (7) \end{aligned}$$

so f is convex (satisfies the required inequality with equality for all $\theta \in [0,1]$).

First Order and Second Order Tests for Convexity

Fact. When f is once differentiable, f is convex if and only if $f(y) \ge f(x) + \nabla f(x)^T(y-x)$ for all x, y.

- The notation $M \succeq 0$ or $M \succ 0$ for a square symmetric matrix M means that M is *positive (semi)definite (PD/PSD)*. Recall that M is PSD (respectively, positive definite) if $x^T M x \ge 0$ (respectively, $x^T M x > 0$ for all x).

Fact. When f is twice differentiable, f is convex if and only if $\nabla^2 f(x) \succeq 0$ for all x.

Example 1:

Consider the quadratic function

$$f(x) = \frac{1}{2}x^T P x + q^T x + r, \qquad P = P^T \tag{8}$$

Then $\nabla^2 f(x) = P$ for all x, so quadratic functions are convex if and only if $P \geq 0$, *i.e.*, P is a positive semidefinite matrix.

Example 2:

Any norm[1] $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is convex:

$$\|\theta x + (1-\theta)y\| \leq \|\theta x\| + \|(1-\theta)y\| = \theta\|x\| + (1-\theta)\|y\| \tag{9}$$

[1] Recall that a norm $\|\cdot\|$ satisfies:

1. $\|x+y\| \leq \|x\| + \|y\|$ (Triangle inequality)
2. $\|ax\| = |a|\|x\|$
3. if $\|x\| = 0$ then $x = 0$

Example 3:

If f is convex, then

$$g(x) = f(Ax+b) \tag{10}$$

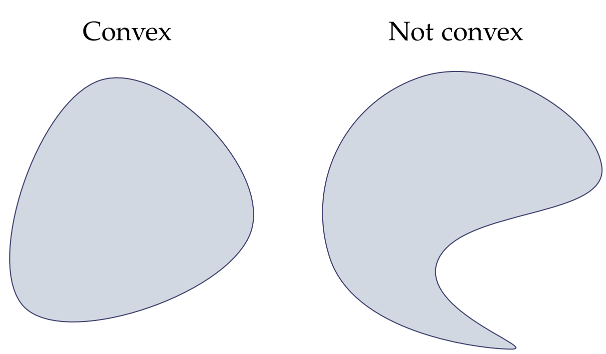
is convex for any A, b:

$$\begin{aligned} g(\theta x + (1-\theta)y) &= f(A(\theta x + (1-\theta)y) + b) && (11)\\ &= f(\theta(Ax+b) + (1-\theta)(Ay+b)) && (12)\\ &\leq \theta f(Ax+b) + (1-\theta)f(Ay+b) && (13)\\ &= \theta g(x) + (1-\theta)g(y). && (14) \end{aligned}$$

Convex Sets

A *convex set* $\mathcal{C}$ satisfies

$$\text{whenever } x_1, x_2 \in \mathcal{C}, \text{ then } \theta x_1 + (1-\theta)x_2 \in \mathcal{C} \text{ for all } 0 \leq \theta \leq 1. \tag{15}$$



Example: Convex Sets as Sublevel Sets of Convex Functions

Any α-*sublevel* set $C_\alpha = \{x : f(x) \leq \alpha\}$ of a convex function is convex.

Proof. Choose $x, y \in C_\alpha$ so that $f(x) \leq \alpha$ and $f(y) \leq \alpha$. By convexity, $f(\theta(x) + (1-\theta)y) \leq \alpha$ for any $0 \leq \theta \leq 1$, and hence $\theta x + (1-\theta)y \in C_\alpha$. □

Note: The converse does *not* hold.

Convex Optimization

Optimization problem from before:

$$\begin{aligned} &\text{minimize} && f_0(x) \\ &\text{subject to} && f_i(x) \leq 0, \qquad i = 1, \ldots, m \end{aligned} \tag{16}$$

The above optimization problem is *convex* if f_0 and all f_i's are convex.

- In this case, the feasible set is a convex set.

Example: Equality Constraints

Convex optimization problems may include equality constraints, but only if they are affine, *i.e.*, of the form $Ax + b = 0$.

Proof. To include an equality constraint $f_i(x) = 0$, we add $f_i(x) \leq 0$ and $-f_i(x) \leq 0$ as inequality constraints. To be convex, we require $f(x)$ and $-f(x)$ to be convex. The only such functions are affine. To see this, we assume f_i is differentiable. Then convexity of f_i and $-f_i$ means:

$$f_i(y) \geq f(x) + \nabla f_i(x)^T(y - x) \tag{17}$$

and

$$-f_i(y) \geq -f_i(x) - \nabla f(x)^T(y - x), \tag{18}$$

so that $f_i(y) = f_i(x) + \nabla f_i(x)^T(y - x)$ for any x, y, *i.e.*, f_i is affine.

□

Theorem: Feasibility of a convex optimization problem. *For a convex optimization problem, a feasible point* x *is optimal if and only if* $\nabla f_0(x)^T(y - x) \geq 0$ *for all feasible* y.

Proof. (if) Since f_0 is convex, for any x, y.

$$f_0(y) \geq f_0(x) + \nabla f_0(x)^T(y - x). \tag{19}$$

Let x be a feasible point such that $\nabla f_0(x)^T(y - x) \geq 0$ for all feasible y. Then for any feasible $y \neq x$, using (19), $f_0(y) \geq f_0(x)$ and x is optimal.

(only if) Now suppose x is optimal but there is some feasible y such that $\nabla f_0(x)^T(y - x) < 0$. The point $z_\theta = \theta y + (1 - \theta)x$ must also be feasible since the feasible set is convex. For small θ, $f(z_\theta) < f(x)$ since $\frac{d}{d\theta} f_0(z_\theta)|_{\theta=0} = \nabla f_0(z_\theta)^T(y - x)\big|_{\theta=0} = \nabla f_0(x)^T(y - x) < 0$.

□

Optimality for Unconstrained Convex Optimization Problems

When all y are feasible, the above condition reduces to: x is optimal if and only if

$$\nabla f_0(x) = 0. \tag{20}$$

Example

Consider

$$\text{minimize}_x \quad \frac{1}{2}x^T P x + q^T x + r \tag{21}$$

where $P \succeq 0$. Then x is optimal if and only if $Px + q = 0$. Three cases:

1. If $q \notin \text{Range}(P)$, no solution. In this case, objective function is unbounded (below)
2. If P is nonsingular (i.e.,$P \succ 0$), then $x^* = -P^{-1}q$ is unique solution
3. If P is singular but $q \in \text{Range}(P)$, then set of optimal points is affine subspace

$$\{x \text{ s.t. } Px = -q\} = \\ \{x^* + y \text{ s.t. } y \in \text{Null}(P),\ x^* \text{ is any vector such that } Px^* = -q\}$$