ME 6402 – Lecture 20

CONTROL LYAPUNOV FUNCTIONS

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Overview:

- Feedback linearization for MIMO Systems
- Define Control Lyapunov Functions
- Present Sontag's Universal Formula for Smooth Stabilization

Additional Reading:

- E. Sontag, 1983
- Z. Arstein, 1978

Multi-Input Multi-Output Systems

Recall that a MIMO system[1] with m inputs and m outputs has individual relative degree r_i for each output (the number of times we need to differente y_i until at least one input appears). Then, the system can be input-output linearized via the control law:

$$u = A(x)^{-1}(-B(x) + v)$$

where A and B are of the form:

$$\begin{bmatrix} y_1^{(r_1)} \\ \vdots \\ y_m^{(r_m)} \end{bmatrix} = \underbrace{\begin{bmatrix} L_f^{r_1} h_1(x) \\ \vdots \\ L_f^{r_m} h_m(x) \end{bmatrix}}_{=:\, B(x)} + \underbrace{\begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & \cdots & L_{g_m} L_f^{r_1-1} h_1(x) \\ \vdots & & \vdots \\ L_{g_1} L_f^{r_m-1} h_m(x) & \cdots & L_{g_m} L_f^{r_m-1} h_m(x) \end{bmatrix}}_{=:\, A(x)} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}.$$

[1] A MIMO system with m inputs and m outputs:

$$\dot{x} = f(x) + \begin{bmatrix} g_1(x) & \dots & g_m(x) \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

$$y_i = h_i(x), \quad i = 1, \cdots, m.$$

This input-output linearizing controller creates m decoupled chains of integrators:

$$y_i^{(r_i)} = v_i, \quad i = 1, \ldots, m.$$

Definition: *Vector Relative Degree*. A system has *vector relative degree* $\{r_1, \cdots, r_m\}$ if the matrix $A(x)$ defined above is nonsingular.

If the system has vector relative degree $\{r_1, \cdots, r_m\}$, then $r := r_1 + \cdots + r_m \leq n$ and the output coordinates of the system are:

$$\eta := [h_1(x)\ L_f h_1(x) \cdots L_f^{r_1-1} h_1(x)\ \cdots\ h_m(x)\ L_f h_m(x) \cdots L_f^{r_m-1} h_m(x)]^T$$

As in normal form discussed in Lecture 17, one can find $n - r$ additional functions $z_1(x), \cdots, z_{n-r}(x)$ so that $x \mapsto (z, \eta)$ is a complete coordinate transformation.

Full-state feedback linearization amounts to finding m output functions $h_1, \cdots, h_m$ such that the system has vector relative degree $\{r_1, \cdots, r_m\}$ with $r_1 + \cdots + r_m = n$. Necessary and sufficient conditions for the existence of such functions, analogous to those in Lecture 18 for SISO systems, are available[2].

[2] see, *e.g.*, Sastry, Proposition 9.16

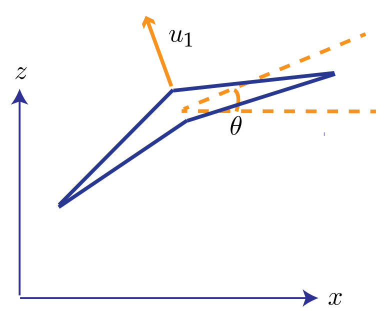
Example:

Consider the following model of a *planar vertical take-off and landing* (PVTOL) aircraft[3]

[3] Sastry, Section 10.4.2

$$\begin{aligned} \ddot{x} &= -\sin(\theta)u_1 + \mu\cos(\theta)u_2 \\ \ddot{z} &= \cos(\theta)u_1 + \mu\sin(\theta)u_2 - 1 \\ \ddot{\theta} &= u_2, \end{aligned}$$

where μ is a constant that accounts for the coupling between the rolling moment and translational acceleration, and -1 in the second equation is the gravitational acceleration, normalized to unity by appropriately scaling the variables.



Taking our state variable to be $\mathbf{x} = [x, \dot{x}, z, \dot{z}, \theta, \dot{\theta}]^T$, we can write the system the control affine form:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{z} \\ \ddot{z} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ 0 \\ \dot{z} \\ -1 \\ \dot{\theta} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -\sin(\theta) & \mu\cos(\theta) \\ 0 & 0 \\ \cos(\theta) & \mu\sin(\theta) \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

If we take x and z as the two outputs we can observe that each output has relative degree 2 ($r_1 = 2$, $r_2 = 2$):

$$\begin{bmatrix} \ddot{x} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \underbrace{\begin{bmatrix} -\sin\theta & \mu\cos\theta \\ \cos\theta & \mu\sin\theta \end{bmatrix}}_{A(\theta)} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

where $A(\theta)$ is invertible when $\mu \neq 0$:

$$A^{-1}(\theta) = \begin{bmatrix} -\sin\theta & \cos\theta \\ \frac{1}{\mu}\cos\theta & \frac{1}{\mu}\sin\theta \end{bmatrix}.$$

Thus the systems has vector relative degree $\{2, 2\}$. This implies that when $\mu \neq 0$, and the input-output linearizing controller is

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -\sin\theta & \cos\theta \\ \frac{1}{\mu}\cos\theta & \frac{1}{\mu}\sin\theta \end{bmatrix} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right).$$

Note, the system will have zero dynamics since $r < n$. Since our output coordinates are aligned with our state variables, we know that the remaining two zero dynamic coordinates will be θ and $\dot{\theta}$. We can solve for the zero dynamics by substituting $u_2^* = \frac{1}{\mu}\sin\theta$ into our dynamics for $\ddot{\theta}$:

$$\ddot{\theta} = \frac{1}{\mu}\sin\theta.$$

The system is nonminimum phase for $\mu > 0$, since $\theta = 0$ is unstable.

Control Lyapunov Functions

Motivation: Feedback linearization stabilizes systems by "cancelling" the nonlinear dynamics and forcing a system to act like a linear one. While this is better than simply "ignoring" nonlinear dynamics (through classic linearization), it still does not take advantage of the natural dynamics of the system. This fundamental limitation is addressed through the use of *control Lyapunov functions*.

Intro: We had previously utilized Lyapunov for *analysis* of the system

$$\dot{x} = f(x), \qquad f(0) = 0 \tag{1}$$

Here, the goal was to find a positive definite Lyapunov function $V(x)$ such that $\dot{V}(x)$ is negative definite to prove asym. stability of $x = 0$.

What about controlling for (asymptotic) stability?

- An idea: For

$$\dot{x} = f(x) + g(x)u \tag{2}$$

  and a candidate positive definite Lyapunov function $V(x)$, choose u such that $\dot{V}$ is negative definite

Definition: *Control Lyapunov Function*. A positive definition function $V(x)$ is a (global) control Lyapunov function (CLF) for (2) if $\forall x \neq 0$, $\exists\, u$ such that

$$\dot{V}(x) = \frac{\partial V}{\partial x}\left[f(x) + g(x)u\right] < 0. \tag{3}$$

Equivalently,

$$\frac{\partial V}{\partial x} g(x) = 0 \quad \text{and} \quad x \neq 0 \quad \Longrightarrow \quad \frac{\partial V}{\partial x} f(x) < 0. \tag{4}$$

In today's lecture we will introduce a closed-form expression for a CLF, known as Sontag's formula. In the next lecture, we will instead solve for u through convex optimization.

If $u \in \mathbb{R}$, Sontag's formula is:

$$u = \phi(x) = \begin{cases} -\left[\left(\frac{\partial V}{\partial x} f\right) + \sqrt{(}\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4)\right] / \left(\frac{\partial V}{\partial x} g\right) & \text{if} \quad \frac{\partial V}{\partial x} g \neq 0 \\ 0 & \text{if} \quad \frac{\partial V}{\partial x} g = 0 \end{cases} \tag{5}$$

Note:

- Choosing $u = \phi(x)$ asymptotically stabilizes the origin (Proof is shown next).
- Formula seems complicate. Why? (Examples shown later)

Proof. Compute $\dot{V}(x)$ for $x \neq 0$:

- If $\frac{\partial V}{\partial x} g(x) = 0$, then

  $$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) < 0$$

  for any $x \neq 0$ by definition of CLF
- If $\frac{\partial V}{\partial x} g(x) \neq 0$, then

  $$\dot{V}(x) = \frac{\partial V}{\partial x}[f(x) + g(x)\phi(x)] = -\sqrt{\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4} < 0$$

  Therefore, $x \neq 0$ implies $\dot{V}(x) < 0$, which shows asymptotic stability.

$\square$

Example 1:

Consider

$$\dot{x} = -x^3 + u$$

with CLF $V(x) = \frac{1}{2}x^2$. Let's consider the following controllers:

1. $u \equiv 0$:
2. Feedback linearizing controller
3. $u = \phi(x)$ from Sontag's formula

Controller 1:

$$\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x} = x(-x^3) = -x^4 < 0 \quad \text{for} \quad x \neq 0 \tag{6}$$

so the system is globally asymptotically stable but not exponentially stable[4].

[4] Recall that exponential stability requires a linear bound on $\dot{V}$ in terms of V itself, i.e.,:

$$\dot{V}(x) \leq -cV(x)$$

for some $c > 0$

Controller 2: With the goal of driving $x \to 0$, we can choose our output $y = x$. This implies that $r = 1$ (since we need to differentiate x once to get to u). The feedback linearizing controller would be

$$u = x^3 + v$$

Therefore, choosing $v = -k_1 x$ for some $k_1 > 0$ yields the closed loop system:

$$\dot{x} = -k_1 x \quad \Longrightarrow \quad \dot{V} = x(-k_1 x) = -k_1 x^2$$

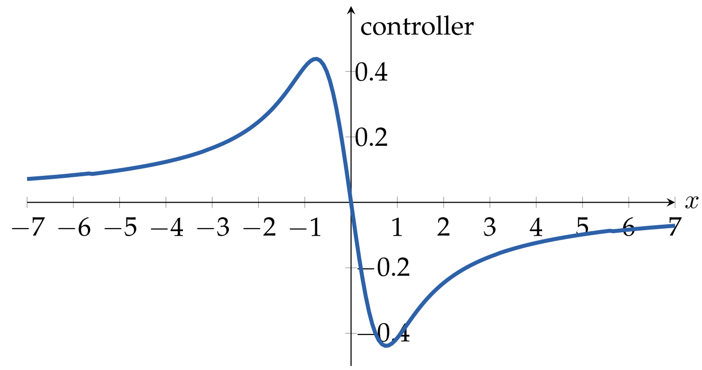
hence the system is now exponentially stable.

Controller 3: To apply Sontag's formula, we need to compute the terms:

$$\frac{\partial V}{\partial x} f(x) = x(-x^3) = -x^4, \quad \frac{\partial V}{\partial x} g(x) = x(1) = x$$

Plugging these into Sontag's formula yields:

$$\begin{aligned} u &= -1/x\left(-x^4 + \sqrt{x^8 + x^4}\right) \\ &= -1/x\left(-x^4 + x^2\sqrt{x^4 + 1}\right) \\ &= x^3 - x\sqrt{x^4 + 1} \end{aligned}$$

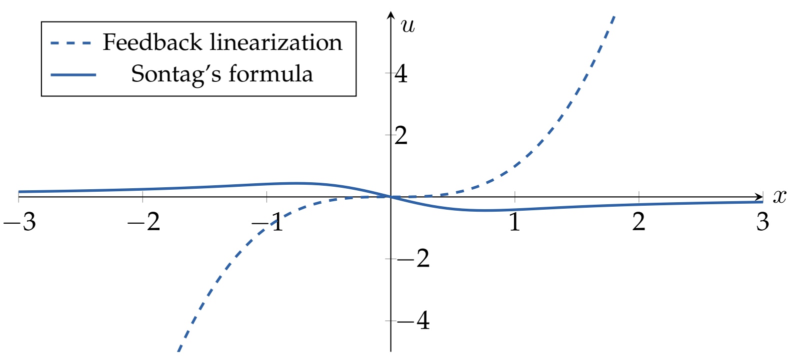


This control law yields the closed-loop system:

$$\dot{V}(x) = x\left(-x^3 + x^3 - x\sqrt{x^4+1}\right) = -x^2\sqrt{x^4+1}$$
$$\leq -x^2$$

where the last inequality follows from the fact that $\sqrt{x^4+1} \geq 1$. Therefore, the system is also globally exponentially stable.

A comparison of the two controllers is shown below:



In general, Sontag's formula can keep useful nonlinearities (like $-x^3$), while feedback linearization cancels all nonlinearities. However, there is no universal theorem that Sontag's formula is always "better".

Example 2:

Consider the system

$$\dot{x} = x - x^3 + u$$

The feedback linearizing controller for this system is:

$$u = -x + x^3 - k_1 x$$

for any $k_1 > 0$.

A CLF (using Sontag's formula) for this system is

$$
\begin{aligned}
u &= \frac{x(-x+x^3) - x\sqrt{x^2(x-x^3)^2 + x^4}}{x} \\
&= -x + x^3 - \sqrt{(1-x^2)^2 + 1}
\end{aligned}
$$

A comparison of the two controllers is shown below: