# *ME 6402 – Lecture 2*[1]

## ESSENTIALLY NONLINEAR PHENOMENA

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Additional Reading:

- Khalil, Chapter 2

Overview

- List several phenomena unique to systems that are not linear
- Phase portraits in the plane

## *Review*

Last class we ended with the pendulum example. Note, we assumed that there was a frictional force resisting the motion that was proportional to the speed of the mass (i.e., $F_{ext} = -kv = -k\ell\dot{\theta}$).

This allowed us to derive the equations of motion:

From the Lagrangian $\mathcal{L}(\theta, \dot{\theta}) = \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\cos\theta$

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} &= \tau_{ext} \\
\frac{d}{dt}\left(m\ell^2\dot{\theta}\right) + mg\ell\sin\theta &= -k\ell^2\dot{\theta} \\
m\ell^2\ddot{\theta} + mg\ell\sin\theta &= -k\ell^2\dot{\theta} \\
\ddot{\theta} + \frac{g}{\ell}\sin\theta &= -\frac{k}{m}\dot{\theta} \\
\ddot{\theta} &= -\frac{k}{m}\dot{\theta} - \frac{g}{\ell}\sin\theta
\end{aligned}$$

From these equations of motion, we can derive a state-space representation of the system by defining the state variables $x_1 = \theta$ and $x_2 = \dot{\theta}$:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{k}{m}x_2 - \frac{g}{\ell}\sin x_1
\end{aligned}$$

or in matrix form:

$$\dot{x} = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_2 - \frac{g}{\ell}\sin x_1 \end{bmatrix}$$

We can identify the equilibrium points of the system by identifying $x^*$ such that $f(x^*) = 0$:

$$x^* = \{(0,0), (\pi, 0)\}$$

Finally, the we can determine the stability of the equilibrium points by linearizing our system around each of the equilibrium points:

$$A = \left.\frac{\partial f}{\partial x}\right|_{x^*} \triangleq \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{x^*}$$

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos x_1 & -\frac{k}{m} \end{bmatrix} \\ J((0,0)) &= \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{k}{m} \end{bmatrix} \quad \longrightarrow \quad \Re(\lambda_i(A)) < 0 \quad \text{(stable)} \\ J((\pi,0)) &= \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & -\frac{k}{m} \end{bmatrix} \quad \longrightarrow \quad \Re(\lambda_1(A)) < 0, \Re(\lambda_2(A)) > 0 \text{(unstable)} \end{aligned}$$

This example demonstrates a situation in which linearization is a valid approach towards analyzing a nonlinear system. As stated in Khalil, "whenever possible, we should make use of linearization to learn as much as we can about the behavior of a nonlinear system". However, linearization has two basic limitations:

- Linearization is only valid in a neighborhood of the equilibrium point ("local approximation").
- Nonlinear system dynamics are much richer than the dynamics of a linear system.

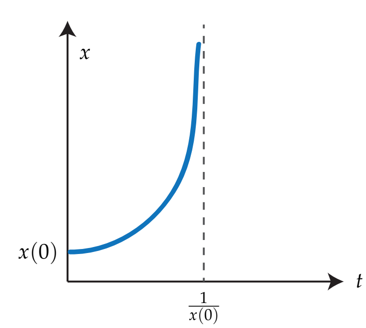
The following are phenomena that can only take place in the presence of nonlinearities and cannot be captured by linear models.

## *Essentially Nonlinear Phenomena*

1. Finite Escape Time: (the state goes to infinity in finite time)
   Example: $\dot{x} = x^2$

$$\begin{aligned} \frac{dx}{dt} &= x^2 \\ \frac{1}{x^2}dx &= dt \\ -\frac{1}{x} &= t + C \\ x(t) &= \frac{1}{C - t} \\ x(t) &= \frac{1}{\frac{1}{x_0} - t} \\ \Rightarrow t_{\text{escape}} &= \frac{1}{x_0} \end{aligned}$$

   For linear systems, $x(t) \to \infty$ cannot happen in finite time.



2. Multiple Isolated Equilibria

   Linear systems: either unique equilibrium or a continuum

   Pendulum: two isolated equilibria (one stable, one unstable)

   "Multi-stable" systems: two or more stable equilibria

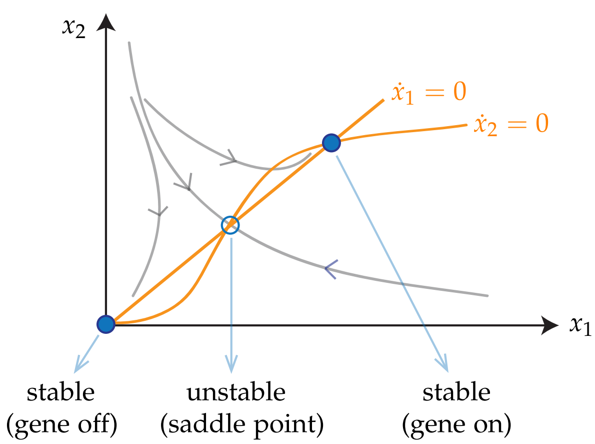
   Example: bistable switch

$$\dot{x}_1 = -ax_1 + x_2 \qquad x_1: \text{ concentration of protein}$$
$$\dot{x}_2 = \frac{x_1^2}{1+x_1^2} - bx_2 \qquad x_2: \text{ concentration of mRNA}$$

   $a > 0$, $b > 0$ are constants. State space: $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$.

   This model describes a positive feedback where the protein encoded by a gene stimulates more transcription via the term $\frac{x_1^2}{1+x_1^2}$.

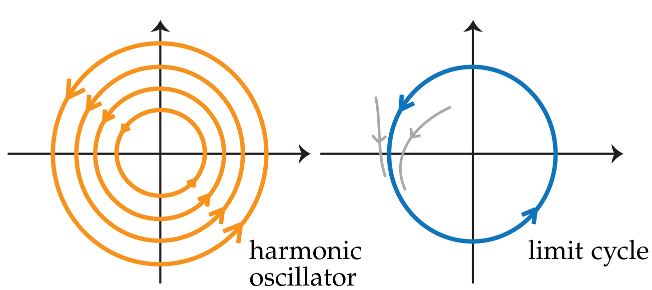
   Single equilibrium at the origin when $ab > 0.5$. If $ab < 0.5$, the line where $\dot{x}_1 = 0$ intersects the sigmoidal curve where $\dot{x}_2 = 0$ at two other points, giving rise to a total of three equilibria:




3. Limit cycles: Linear oscillators exhibit a continuum of periodic orbits; *e.g.*, every circle is a periodic orbit for $\dot{x} = Ax$ where

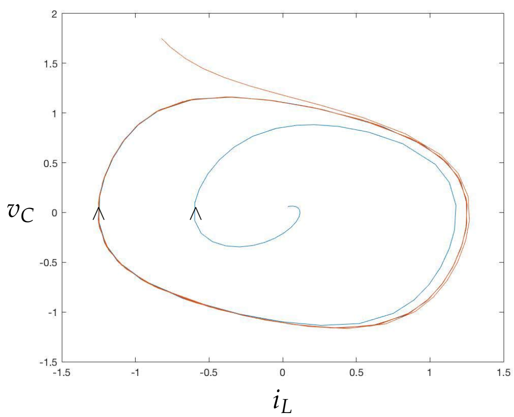
$$A = \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix} \qquad (\lambda_{1,2} = \mp j\beta).$$

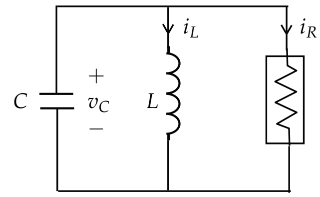
   In contrast, a limit cycle is an isolated periodic orbit and can occur only in nonlinear systems.

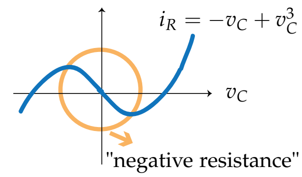


Example: van der Pol oscillator (models a system with self-sustained oscillations)

$$C\dot{v}_C = -i_L + v_C - v_C^3$$
$$L\dot{i}_L = v_C$$



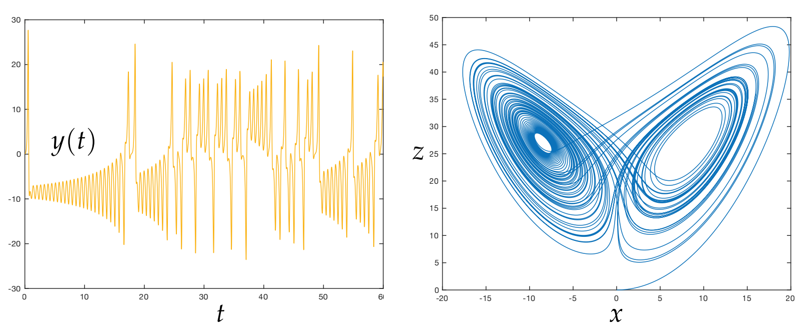




4. Chaos: Irregular oscillations, never exactly repeating.

Example: Lorenz system (derived by Ed Lorenz in 1963 as a simplified model of convection rolls in the atmosphere):

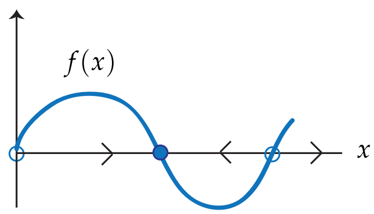
$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz.$$

Chaotic behavior with $\sigma = 10,\ b = 8/3,\ r = 28$:




- For continuous-time, time-invariant systems, $n \geq 3$ state variables required for chaos.

  $n = 1$: $x(t)$ monotone in $t$, no oscillations:

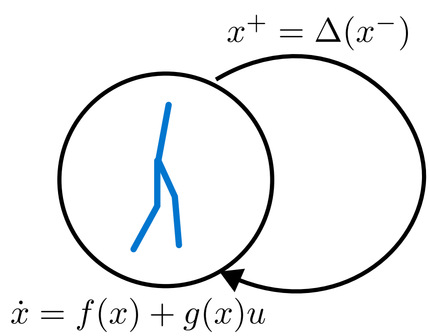



$\underline{n = 2}$: Poincaré-Bendixson Theorem (to be studied in Lecture 4) guarantees regular behavior.

- Poincaré-Bendixson does not apply to time-varying systems and $n \geq 2$ is enough for chaos (for Van der Pol oscillator can exhibit chaos).
- For discrete-time systems, $n = 1$ is enough (we will see an example in Lecture 6).

5. Multiple modes of behavior:

   Hybrid systems exhibit both continuous and discrete dynamics. Examples include a bouncing ball, or a legged robot.



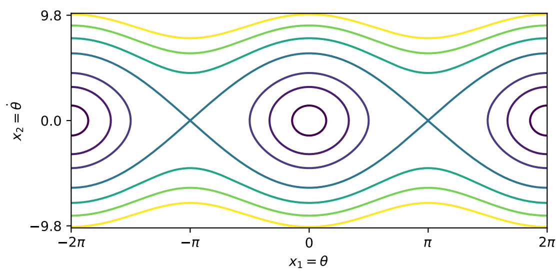

## *Planar (Second Order) Dynamical Systems*

Chapter 2 in both Sastry and Khalil

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

Second-order autonomous systems are convenient to study because solution trajectories (i.e., $x(t) = x_1(t), x_2(t)$) can be represented as curves in the plane. This "plane" is usually called the *phase plane* with $f(x)$ the *vector field* on the phase plane.

The family of all trajectories (solution curves) is called the *phase portrait* of the system. For example, recall that the phase portrait of the pendulum system was:




*Phase Portraits of Linear Systems:* $\dot{x} = Ax$

Depending on the eigenvalues of $A$, the real Jordan form $J = T^{-1}AT$ has one of three forms:

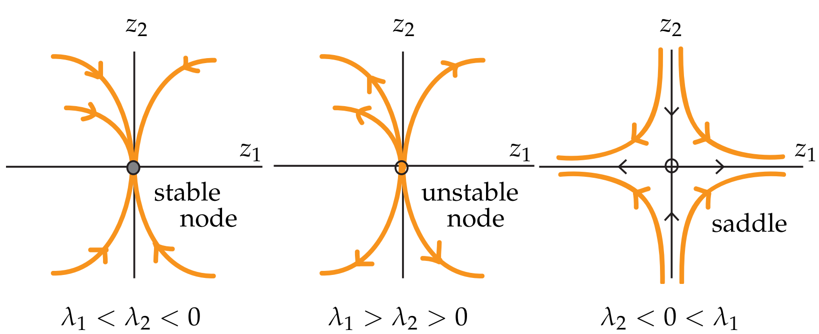
- Distinct real eigenvalues

$$T^{-1}AT = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

  In $z = T^{-1}x$ coordinates:

$$\dot{z}_1 = \lambda_1 z_1, \quad \dot{z}_2 = \lambda_2 z_2.$$

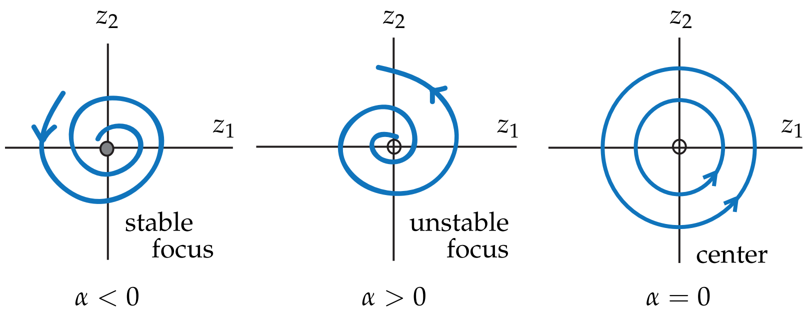
  The equilibrium is called a *node* when $\lambda_1$ and $\lambda_2$ have the same sign (*stable* node when negative and *unstable* when positive). It is called a *saddle point* when $\lambda_1$ and $\lambda_2$ have opposite signs.




- Complex eigenvalues: $\lambda_{1,2} = \alpha \mp j\beta$

$$T^{-1}AT = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

$$\begin{aligned} \dot{z}_1 &= \alpha z_1 - \beta z_2 \\ \dot{z}_2 &= \alpha z_2 + \beta z_1 \end{aligned} \quad \rightarrow \quad \text{polar coordinates} \quad \rightarrow \quad \begin{aligned} \dot{r} &= \alpha r \\ \dot{\theta} &= \beta \end{aligned}$$




The phase portraits above assume $\beta > 0$ so that the direction of rotation is counter-clockwise: $\dot{\theta} = \beta > 0$.