ME 6402 – Lecture 18

FEEDBACK LINEARIZATION 3 (FULL-STATE FEEDBACK LINEARIZATION)

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Overview:

- Introduce full-state feedback linearization
- Define a few basic concepts from differential geometry
- Frobenius Theorem

Additional Reading:

- Khalil, Chapter 13
- Sastry, Chapter 9

Review of Normal Form

To find the zero dynamics coordinates z, we must find $n - r$ independent variables z such that $\dot{z}$ does not contain u. An alternative approach is to find coordinates z_i (with $i = 1, \ldots, n - r$) such that:

$$\frac{\partial z_i}{\partial x} g(x) = \left[\frac{\partial z_i}{\partial x_1} \cdots \frac{\partial z_i}{x_n} \right] g(x) \equiv 0$$

This can be interpreted as the coordinates z being chosen to be *normal* to the actuation vector $g(x)$.

Example

Consider the system

$$\begin{aligned}
\dot{x}_1 &= -x_1 + \frac{2 + x_3^2}{1 + x_3^2} u \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_1 x_3 + u \\
y &= x_2
\end{aligned}$$

We can inspect the relative degree of the system by observing:

$$\begin{aligned}
y &= x_2 \\
\dot{y} &= \dot{x}_2 = x_3 \\
\ddot{y} &= \dot{x}_3 = x_1 x_3 + u
\end{aligned}$$

Thus, the system has relative degree $r = 2$. Since $r < n$ (and our system is SISO), we will have one degree of underactuation. To characterize the zero dynamics, we restrict x to:

$$Z = \{x \in \mathbb{R}^3 \text{ s.t. } x_2 = x_3 = 0\}$$

and take $v = 0$ for the input-output linearizing control law:

$$\begin{aligned} u^*(x) &= \frac{1}{L_g L_f h(x)} \left(-L_f^2 h(x) + v\right) \\ &= -x_1 x_3 + v \\ \implies u &= -x_1 x_3 \end{aligned}$$

This yields the zero dynamics:

$$\begin{aligned} \dot{x}_1 &= -x_1 + \frac{2 + x_3^2}{1 + x_3^2} \underbrace{(-x_1 x_3)}_{0} = -x_1 \\ \dot{x}_2 &= \underbrace{x_3}_{0} = 0 \\ \dot{x}_3 &= x_1 x_3 - x_1 x_3 = 0 \end{aligned}$$

Thus, the zero dynamics are $\dot{x}_1 = -x_1$.

Finally, to find the zero dynamics coordinate z for normal form, we need to find a function $z(x)$ such that

$$z(0) = 0, \quad \frac{\partial z}{\partial x} g(x) = 0$$

Since $g(x) = [\frac{2+x_3^2}{1+x_3^2}, 0, 1]^T$:

$$\frac{\partial z_i}{\partial x} g(x) = 0 \implies \frac{\partial z}{\partial x_1} \cdot \frac{2 + x_3^2}{1 + x_3^2} + \frac{\partial z}{\partial x_3} = 0$$

This partial differential equation can be solved by separating variables to obtain

$$z(x) = -x_1 + x_3 + \tan^{-1}(x_3)$$

Thus, the normal form is:

$$T(x) = \begin{bmatrix} z \\ \zeta_1 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} z \\ \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} -x_1 + x_3 + \tan^{-1}(x_3) \\ x_2 \\ x_3 \end{bmatrix}$$

NOTE: From here on out I will be denoting the output coordinates as η instead of ζ since it is difficult for me to write ζ...

Full-State Feedback Linearization

The system $\dot{x} = f(x) + g(x)u$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, is (full state) feedback linearizable if a function $h : \mathbb{R}^n \mapsto \mathbb{R}$ exists such that the relative degree from u to $y = h(x)$ is n.

Since $r = n$, the normal form in Lecture 19 has no zero dynamics and

$$x \to \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{bmatrix} = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix}$$

is a diffeomorphism that transforms the system to the form:

$$\begin{aligned} \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= \eta_3 \\ &\vdots \\ \dot{\eta}_n &= L_f^n h(x) + L_g L_f^{n-1} h(x) u. \end{aligned}$$

Then, the feedback linearizing controller

$$u = \frac{1}{L_g L_f^{n-1} h(x)} \Big(- L_f^n h(x) + v \Big), \quad v = -k_1 \eta_1 \cdots - k_n \eta_n,$$

yields the closed-loop system:

$$\dot{\eta} = A\eta \quad \text{where} \quad A = \begin{bmatrix} 0 & 1 & 0 & \dots & \\ 0 & 0 & 1 & \dots & \\ & & & \ddots & \\ & & & & 1 \\ -k_1 & -k_2 & -k_3 & \dots & -k_n \end{bmatrix}.$$

Example

Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 + 2x_1^2 \\ \dot{x}_2 &= x_3 + u \\ \dot{x}_3 &= x_1 - x_3 \end{aligned}$$

The choice $y = x_3$ gives relative degree $r = n = 3$. Thus, we know that the normal form will be of the form:

$$T(x) = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} h(x) \\ L_f h(x) \\ L_f^2 h(x) \end{bmatrix} = \begin{bmatrix} x_3 \\ x_1 - x_3 \\ x_2 + 2x_1^2 - (x_1 - x_3) \end{bmatrix}$$

This gives us our strict feedback form:

$$\dot{\eta} = \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{\eta}_3 \end{bmatrix} = \begin{bmatrix} \eta_2 \\ \eta_3 \\ (4x_1 - 1)(x_2 + 2x_1^2) + x_1 + u \end{bmatrix}$$

with the feedback linearizing controller:

$$u = -(4x_1 - 1)(x_2 + 2x_1^2) - x_1 - k_1\eta_1 - k_2\eta_2 - k_3\eta_3.$$

Summary so far:

I/O Linearization:	• suitable for tracking • output y is an intrinsic physical variable
Full state linearization:	• set point stabilization • output is not intrinsic, selected to enable a linearizing change of variables.

Remaining question:

- When is a system feedback linearizable, *i.e.*, how do we know whether a relative degree $r = n$ output exists?

Basic Definitions from Differential Geometry

Definition: The Lie bracket of two vector fields f and g is a new vector field defined as:

$$[f, g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x).$$

Note:

1. $[f, g] = -[g, f]$,
2. $[f, f] = 0$,
3. If f, g are constant then $[f, g] = 0$.

Notation for repeated applications:

$$[f, [f, g]] = \mathrm{ad}_f^2\, g, \quad [f, [f, [f, g]]] = \mathrm{ad}_f^3\, g, \quad \cdots$$

$$\mathrm{ad}_f^0\, g(x) \triangleq g(x), \quad \mathrm{ad}_f^k\, g \triangleq [f, \mathrm{ad}_f^{k-1}\, g] \quad k = 1, 2, 3, \ldots$$

Definition: Given vector fields $f_1, \ldots, f_k$, a distribution Δ is defined as $\Delta(x) = \mathrm{span}\{f_1(x), \ldots, f_k(x)\}$.

$f \in \Delta$ means that there exist scalar functions $\alpha_i(x)$ such that

$$f(x) = \alpha_1(x) f_1(x) + \cdots + \alpha_k(x) f_k(x).$$

Definition: Δ is said to be nonsingular if $f_1(x), \ldots, f_k(x)$ are linearly independent for all x.

Definition: Δ is said to be involutive if

$$g_1 \in \Delta, g_2 \in \Delta \implies [g_1, g_2] \in \Delta$$

that is, Δ is closed under the Lie bracket operation.

Proposition: $\Delta = \text{span}\{f_1, \ldots, f_k\}$ is involutive if and only if

$$[f_i, f_j] \in \Delta \quad 1 \le i, j \le k.$$

Example 1: $\Delta = \text{span}\{f_1, \ldots, f_k\}$ where $f_1, \ldots, f_k$ are constant vectors

Example 2: a single vector field $f(x)$ is involutive since $[f, f] = 0 \in \Delta$

Definition: A nonsingular k-dimensional distribution

$$\Delta(x) = \text{span}\{f_1(x), \ldots, f_k(x)\} \quad x \in \mathbb{R}^n$$

is said to be completely integrable if there exist $n-k$ functions

$$\phi_1(x), \ldots, \phi_{n-k}(x)$$

such that

$$\frac{\partial \phi_i}{\partial x} f_j(x) = 0 \quad i = 1, \ldots, n-k, \quad j = 1, \ldots, k$$

and $d\Phi_i(x) := \dfrac{\partial \phi_i}{\partial x}$, $i = 1, \ldots, n-k$, are linearly independent.

Frobenius Theorem: A nonsingular distribution is completely integrable if and only if it is involutive.

Back to (Full State) Feedback Linearization

Recall: $\dot{x} = f(x) + g(x)u$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ is feedback linearizable if we can find an output $y = h(x)$ such that relative degree $r = n$.

How do we determine if a relative degree $r = n$ output exists?

$$L_g h(x) = L_g L_f h(x) = \cdots = L_g L_f^{n-2} h(x) = 0 \text{ in a nbhd of } x_0 \quad (1)$$

$$L_g L_f^{n-1} h(x_0) \neq 0. \quad (2)$$

Proposition:[1] (1)-(2) are equivalent to:

[1] follows from (5) below with $j = 0$

$$L_g h(x) = L_{\text{ad}_f g} h(x) = \cdots = L_{\text{ad}_f^{n-2} g} h(x) = 0 \text{ in a nbhd of } x_0 \quad (3)$$

$$L_{\text{ad}_f^{n-1} g} h(x_0) \neq 0. \quad (4)$$

The advantage of (3) over (1) is that it has the form:

$$\frac{\partial h}{\partial x}[g(x) \quad \mathrm{ad}_f\, g(x) \quad \dots \quad \mathrm{ad}_f^{n-2} g(x)] = 0$$

which is amenable to the Frobenius Theorem.

Theorem: $\dot{x} = f(x) + g(x)u$ is feedback linearizable around x_0 if and only if the following two conditions hold:

C1) $[g(x_0) \quad \mathrm{ad}_f\, g(x_0) \quad \dots \quad \mathrm{ad}_f^{n-1} g(x_0)]$ has rank n

C2) $\Delta(x) = \mathrm{span}\{g(x), \mathrm{ad}_f\, g(x), \dots, \mathrm{ad}_f^{n-2} g(x)\}$ is involutive in a neighborhood of x_0.

In practice, we use the Frobenius Theorem to show that the distribution $\Delta(x)$ is involutive by finding a function $\phi(x)$ such that $\frac{\partial \phi}{\partial x}[g(x), \mathrm{ad}_f\, g(x), \dots, \mathrm{ad}_f^{n-2} g(x)] \equiv 0$.

Proof: (if) Given C1 and C2 show that there exists $h(x)$ satisfying (3)-(4).

$\Delta(x)$ is nonsingular by C1 and involutive by C2. Thus, by the Frobenius Theorem, there exists $h(x)$ satisfying (3) and $dh(x) \neq 0$.

To prove (4) suppose, to the contrary, $L_{\mathrm{ad}_f^{n-1}} h(x_0) = 0$. This implies

$$dh(x_0)\underbrace{[g(x_0) \quad \mathrm{ad}_f\, g(x_0) \quad \dots \quad \mathrm{ad}_f^{n-1} g(x_0)]}_{\text{nonsingular by C1}} = 0.$$

Thus $dh(x_0) = 0$, a contradiction.

(only if) Given that $y = h(x)$ with $r = n$ exists, that is (3)-(4) hold, show that C1 and C2 are true.

We will use the following fact[2] which holds when $r = n$:

[2] see, *e.g.*, Khalil, Lemma C.8

$$L_{\mathrm{ad}_f^i g} L_f^j h(x) = \begin{cases} 0 & \text{if } i + j \leq n-2 \\ (-1)^{n-1-j} L_g L_f^{n-1} h(x) \neq 0 & \text{if } i + j = n-1. \end{cases} \tag{5}$$

Define the matrix

$$M = \begin{bmatrix} dh \\ dL_f h \\ \vdots \\ dL_f^{n-1} h \end{bmatrix} \begin{bmatrix} g & -\mathrm{ad}_f\, g & \mathrm{ad}_f^2\, g & \dots & (-1)^{n-1}\, \mathrm{ad}_f^{n-1} g \end{bmatrix} \tag{6}$$

and note that the (k, ℓ) entry is:

$$\begin{aligned} M_{k\ell} &= dL_f^{k-1} h(x) (-1)^{\ell-1}\, \mathrm{ad}_f^{\ell-1}\, g(x) \\ &= (-1)^{\ell-1} L_{\mathrm{ad}_f^{\ell-1} g} L_f^{k-1} h(x). \end{aligned}$$

Then, from (5):

$$M_{k\ell} = \begin{cases} 0 & \ell + k \leq n \\ \neq 0 & \ell + k = n+1. \end{cases}$$

$$\begin{bmatrix} 0 & 0 & \cdots & \star \\ 0 & & / & \vdots \\ \vdots & \star & & \vdots \\ \star & \cdots & \cdots & \star \end{bmatrix}$$

Since the diagonal entries are nonzero, M has rank n and thus the factor

$$\begin{bmatrix} g & -\text{ad}_f\, g & \text{ad}_f^2\, g & \dots & (-1)^{n-1}\,\text{ad}_f^{n-1} g \end{bmatrix}$$

in (6) must have rank n as well. Thus C1 follows.

This also implies $\Delta(x)$ is nonsingular; thus, by the Frobenius Thm,

$$\text{complete integrability} \;\equiv\; \text{involutivity.}$$

$\Delta(x)$ is completely integrable since $h(x)$ satisfying (3) exists by assumption; thus, we conclude involutivity (C2). □

<u>Example:</u>

$$\begin{aligned} \dot{x}_1 &= x_2 + 2x_1^2 \\ \dot{x}_2 &= x_3 + u \\ \dot{x}_3 &= x_1 - x_3 \end{aligned}$$

Feedback linearizability was shown on page 1 by inspection: $y = x_3$ gives relative degree $= 3$. Verify with the theorem above:

Recall that $[f, g] = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x)$

$$[f, g] = 0 - \begin{bmatrix} 4x_1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$[f, [f, g]] = 0 - \begin{bmatrix} 4x_1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 0 \\ 1 \end{bmatrix}$$

$$f(x) = \begin{bmatrix} x_2 + 2x_1^2 \\ x_3 \\ x_1 - x_3 \end{bmatrix} \qquad g(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$[f, g](x) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \qquad [f, [f, g]](x) = \begin{bmatrix} 4x_1 \\ 0 \\ 1 \end{bmatrix}$$

Conditions of the theorem:

1. $\begin{bmatrix} 0 & -1 & 4x_1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ full rank

2. $\Delta = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right\}$ involutive

   $\dfrac{\partial h}{\partial x} \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$ satisfied by $h(x) = x_3$.