ME 6402 – Lecture 17

FEEDBACK LINEARIZATION 2

March 6 2025

Overview:

- Normal form for input-output feedback linearizable systems

Additional Reading:

- Khalil 13.2

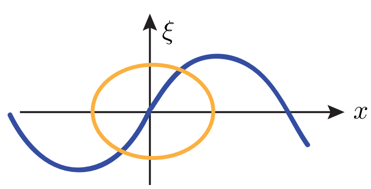
Feedback Linearization (continued)

Nonlinear Changes of Variables

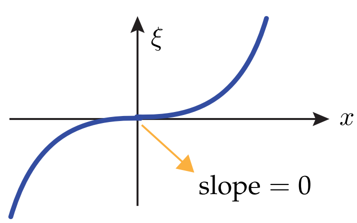
$T : \mathbb{R}^n \to \mathbb{R}^n$ is called a *diffeomorphism* if its inverse T^{-1} exists, and both T and T^{-1} are continuously differentiable (C^1).

Examples:

1. $\xi = Tx$ is a diffeomorphism if T is a nonsingular matrix
2. $\xi = \sin x$ is a local diffeomorphism around $x = 0$, but not global



3. $\xi = x^3$ is not a diffeomorphism because $T^{-1}(\cdot)$ is not C^1 at $\xi = 0$



How to check if $\xi = T(x)$ is a local diffeomorphism?

Implicit Function Theorem[1]

Suppose $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is C^1 and there exists $x_0 \in \mathbb{R}^n$, $\xi_0 \in \mathbb{R}^n$ such that

$$f(x_0, \xi_0) = 0.$$

[1] The key idea is that if a function has a regular Jacobian, then it locally behaves like a bijection with a smooth inverse

If $\frac{\partial f}{\partial x}(x_0, \xi_0)$ is nonsingular, then in a neighborhood of (x_0, ξ_0),

$$f(x, \xi) = 0$$

has a unique solution $x = g(\xi)$ where g is C^1 at $\xi = \xi_0$.

Corollary: Let $f(x, \xi) = T(x) - \xi$. If $\frac{\partial T}{\partial x}$ is nonsingular at x_0, then $T(\cdot)$ is a local diffeomorphism around x_0.

A "Normal Form" that Explicitly Displays the Zero Dynamics

Theorem: If $\dot{x} = f(x) + g(x)u$, $y = h(x)$ has a well-defined relative degree $r \leq n$, then there exist a diffeomorphism $T : x \mapsto \begin{bmatrix} z \\ \zeta \end{bmatrix}$, $z \in \mathbb{R}^{n-r}$, $\zeta \in \mathbb{R}^r$, that transforms the system to the form:

$$\begin{aligned} \dot{z} &= f_0(z, \zeta) \\ \dot{\zeta}_1 &= \zeta_2 \\ &\vdots \\ \dot{\zeta}_r &= b(z, \zeta) + a(z, \zeta)u, \quad y = \zeta_1. \end{aligned} \tag{1}$$

In particular, $\dot{z} = f_0(z, 0)$ represents the zero dynamics. $\square$

To obtain this form, let $\zeta = [h(x) \quad L_f h(x) \quad \ldots \quad L_f^{r-1} h(x)]^T$, and find $n - r$ independent variables z such that $\dot{z}$ does not contain u.

Note that the terms $b(z, \zeta)$ and $a(z, \zeta)$ correspond to $L_f^r(x)$ and $L_g L_f^{r-1} h(x)$ in the original coordinates.

In summary, this change of coordinates is defined by the map:

$$T : x \mapsto \begin{bmatrix} z_1(x) \\ \vdots \\ z_{n-r}(x) \\ \zeta_1(x) \\ \zeta_2(x) \\ \vdots \\ \zeta_r(x) \end{bmatrix} = \begin{bmatrix} z_1(x) \\ \vdots \\ z_{n-r}(x) \\ h(x) \\ L_f h(x) \\ \vdots \\ L_f^{r-1} h(x) \end{bmatrix}, \quad DT = \begin{bmatrix} \frac{\partial z_1}{\partial x_1} & \cdots & \frac{\partial z_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial z_{n-r}}{\partial x_1} & \cdots & \frac{\partial z_{n-r}}{\partial x_n} \\ \frac{\partial h}{\partial x_1} & \cdots & \frac{\partial h}{\partial x_n} \\ \frac{\partial L_f h}{\partial x_1} & \cdots & \frac{\partial L_f h}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial L_f^{r-1} h}{\partial x_1} & \cdots & \frac{\partial L_f^{r-1} h}{\partial x_n} \end{bmatrix} \tag{2}$$

- This map is a diffeomorphism if its Jacobian (DT) has full rank.
- Given a SISO system with a well-defined relative degree r. Then there exist coordinates z_i, $i = 1, \ldots, n - r$ such that the map $T : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism.

Example:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \alpha x_3 + u \\ \dot{x}_3 &= \beta x_3 - u \\ y &= x_1.\end{aligned}$$

Let $\zeta_1 = x_1$, $\zeta_2 = x_2$, and note that $z = x_2 + x_3$ is independent of ζ_1, ζ_2, and $\dot{z}$ does not contain u. Thus, the normal form is:

$$\begin{aligned}\dot{z} &= (\alpha+\beta)x_3 = (\alpha+\beta)z - (\alpha+\beta)\zeta_2 \\ \dot{\zeta}_1 &= \zeta_2 \\ \dot{\zeta}_2 &= \alpha x_3 + u = \alpha z - \alpha\zeta_2 + u.\end{aligned}$$

I/O Linearizing Controller in the new coordinates (1):

$$u = \frac{1}{a(z,\zeta)}\Big(-b(z,\zeta) + v\Big) \tag{3}$$

$$v = -k_1\zeta_1 \cdots - k_r\zeta_r \tag{4}$$

where $k_1, \cdots, k_r$ are such that all roots of $s^r + k_r s^{r-1} + \cdots + k_2 s + k_1$ have negative real parts.

Theorem: If $z = 0$ is locally exponentially stable for the zero dynamics $\dot{z} = f_0(z, 0)$, then (3)–(4) locally exponentially stabilizes $x = 0$.

Proof: Closed-loop system:

$$\begin{aligned}\dot{z} &= f_0(z,\zeta) \\ \dot{\zeta} &= A\zeta\end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & \\ 0 & 0 & 1 & \dots & \\ & & & \ddots & \\ & & & & 1 \\ -k_1 & -k_2 & -k_3 & \dots & -k_r \end{bmatrix}$$

is Hurwitz. The Jacobian linearization at $(z,\zeta) = 0$ is:

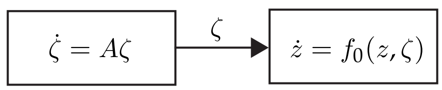
$$J = \begin{bmatrix} \frac{\partial f_0}{\partial z}(0,0) & \frac{\partial f_0}{\partial \zeta}(0,0) \\ 0 & A \end{bmatrix}$$

where $\frac{\partial f_0}{\partial z}(0,0)$ is Hurwitz since $\dot{z} = f_0(z,0)$ is exponentially stable by the proposition in Lecture 12, page 2. Since A is also Hurwitz, all eigenvalues of J have negative real parts $\Rightarrow$ exponential stability.

Global asymptotic stability can be guaranteed with additional assumptions on the zero dynamics, such as ISS of

$$\dot{z} = f_0(z, \zeta)$$

with respect to the input ξ:



<u>Example:</u> $\dot{z} = -z + z^2\zeta, \quad \dot{\zeta} = -k\zeta$

Note: the z subsystem is not ISS

$(z, \zeta) = 0$ is locally exponentially stable, but not globally: solutions escape in finite time for large $z(0)$.

I/O Linearizing Controller for Tracking

For the output $y(t)$ to track a reference signal[2] $y_d(t)$, replace (4) with:

$$v = -k_1(\zeta_1 - y_d(t)) - k_2(\zeta_2 - \dot{y}_d(t)) \cdots - k_r(\zeta_r - y_d^{(r-1)}(t)) + y_d^{(r)}(t)$$

[2] assumed to be r times differentiable

Let $e_1 \triangleq \zeta_1 - y_d(t)$, $e_2 \triangleq \zeta_2 - \dot{y}_d(t)$, $\ldots$, $e_r \triangleq \zeta_r - y_d^{(r-1)}(t)$. Then:

$$\left.\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= e_3 \\ &\vdots \\ \dot{e}_r &= v - y_d^{(r)}(t) = -k_1 e_1 - \cdots - k_r e_r \end{aligned}\right\} \quad \dot{e} = Ae.$$

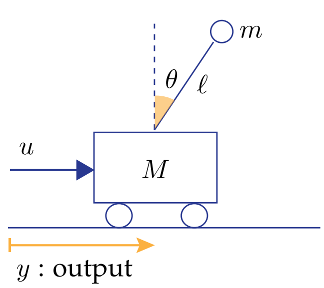
Thus $e(t) \to 0$, that is $y(t) - y_d(t) \to 0$.

If $y_d(t)$ and its derivatives are bounded, then $\zeta(t)$ is bounded. If the zero dynamics $\dot{z} = f_0(z, \zeta)$ is ISS with respect to ζ, then $z(t)$ is also bounded. Thus, all internal signals are bounded.

Cart-Pole Revisited

<u>Example:</u> Cart/Pole [3]

[3] Simulation code for this example is available online



$$
\begin{aligned}
\ddot{y} &= \frac{1}{\frac{M}{m} + \sin^2\theta}\left(\frac{u}{m} + \dot{\theta}^2 \ell \sin\theta - g\sin\theta\cos\theta\right) \\
\ddot{\theta} &= \frac{1}{\ell(\frac{M}{m} + \sin^2\theta)}\left(-\frac{u}{m}\cos\theta - \dot{\theta}^2\ell\cos\theta\sin\theta + \frac{M+m}{m} g\sin\theta\right)
\end{aligned} \tag{5}
$$

In a bit more detail, if we select our system state to be $x = (p, \dot{p}, \theta, \dot{\theta})$, with p being the horizontal position of the cart, then our output is $y = p$. The system dynamics can be explicitly written as:

$$
\dot{x} = \begin{bmatrix} \dot{p} \\ \ddot{p} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{\frac{M}{m}+\sin^2\theta}\left(\frac{u}{m} + \dot{\theta}^2\ell\sin\theta - g\sin\theta\cos\theta\right) \\ x_4 \\ \frac{1}{\ell(\frac{M}{m}+\sin^2\theta)}\left(-\frac{u}{m}\cos\theta - \dot{\theta}^2\ell\cos\theta\sin\theta + \frac{M+m}{m}g\sin\theta\right) \end{bmatrix}
$$

The relative degree of the system is 2, as we can see by differentiating the output twice with respect to time:

$$
\begin{aligned}
y &= p \\
\dot{y} &= \dot{p} \\
\ddot{y} &= \frac{1}{\frac{M}{m} + \sin^2\theta}\left(\frac{u}{m} + \dot{\theta}^2\ell\sin\theta - g\sin\theta\cos\theta\right)
\end{aligned}
$$

The input-output linearizing controller for this system would thus be:

$$
u = -m\left(\dot{\theta}^2\ell\sin\theta - g\sin\theta\cos\theta + v\left(\frac{M}{m} + \sin^2\theta\right)\right)
$$

where $v = -k_1(p - p_d) - k_2(\dot{p} - \dot{p}_d)$.

As we showed in last class, we can find the zero dynamics, by substituting $y = \dot{y} = 0$, and

$$
u^* = -m(\dot{\theta}^2\ell\sin\theta - g\sin\theta\cos\theta)
$$

in the $\ddot{\theta}$ equation:

$$
\boxed{\ddot{\theta} = \frac{g}{\ell}\sin\theta.}
$$

Since the zero dynamics and output dynamics of our system are decoupled completely, the normal form can be written using the transformation:

$$
T : x \mapsto \begin{bmatrix} z_1 \\ z_2 \\ \zeta_1 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ x_1 \\ x_2 \end{bmatrix}
$$