ME 6402 – Lecture 16

FEEDBACK LINEARIZATION 1

March 4 2025

Overview:

- Relative Degree
- Input-Output Linearization
- Zero Dynamics

Additional Reading

- Khalil 13.2

Relative Degree

Consider the single-input single-output (SISO) nonlinear system:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x). \end{aligned} \tag{1}$$

with $y \in \mathbb{R}$ and $u \in \mathbb{R}$.

Relative degree (informal definition): Number of times we need to take the time derivative of the output to see the input:

$L_f h$ is called the *Lie derivative* of h along the vector field f

$$\begin{aligned} \dot{y} &= \frac{dh(x)}{dt} \\ &= \frac{\partial h(x)}{\partial x}\frac{\partial x}{\partial t} \\ &= \underbrace{\frac{\partial h}{\partial x}f(x)}_{=:\, L_f h(x)} + \underbrace{\frac{\partial h}{\partial x}g(x)}_{=:\, L_g h(x)} u \end{aligned}$$

If $L_g h(x) \neq 0$ in an open set containing the equilibrium, then the relative degree is equal to 1. If $L_g h(x) \equiv 0$, continue taking derivatives:

$$\ddot{y} = \underbrace{L_f L_f h(x)}_{=:\, L_f^2 h(x)} + L_g L_f h(x) u.$$

If $L_g L_f h(x) \neq 0$, then relative degree is 2. If $L_g L_f h(x) \equiv 0$, continue.

Definition: *Relative Degree.* The system (1) has *relative degree* r if, in a neighbourhood of the equilibrium,

$$\begin{aligned} L_g L_f^{i-1} h(x) &= 0 \quad i = 1, 2, \ldots, r-1 \\ L_g L_f^{r-1} h(x) &\neq 0. \end{aligned} \tag{2}$$

Examples:

1.
$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1^3 + u\\ y &= x_1\end{aligned} \tag{3}$$

has relative degree = 2 as shown below:

$$\begin{aligned}y &= x_1\\ \dot{y} &= \dot{x}_1 = x_2\\ \ddot{y} &= \dot{x}_2 = -x_1^3 + u\end{aligned}$$

2. SISO linear system:

$$\dot{x} = Ax + Bu \quad y = Cx$$

$L_g h(x) = CB,\ \ L_g L_f h(x) = CAB,\ \ \dots,\ \ L_g L_f^{r-1} = CA^{r-1}B.$

$CB \neq 0 \ \Rightarrow$ relative degree = 1

$CB = 0,\ \ CAB \neq 0 \ \Rightarrow$ relative degree = 2

$CB = \dots = CA^{r-2}B = 0,\ \ CA^{r-1}B \neq 0 \ \Rightarrow$ relative degree = r

The parameters $CA^{i-1}B \ \ i = 1, 2, 3, \dots$ are called *Markov parameters* and are invariant under similarity transformations.

3.
$$\begin{aligned}\dot{x}_1 &= x_2 + x_3^3 \qquad & y &= x_1\\ \dot{x}_2 &= x_3 & \dot{y} &= \dot{x}_1 = x_2 + x_3^3\\ \dot{x}_3 &= u & \ddot{y} &= \dot{x}_2 + 3x_3^2\dot{x}_3 = x_3 + 3x_3^2 u\end{aligned}$$

$L_g L_f h(x) = 3x_3^2 = 0$ when $x_3 = 0$, and $\neq 0$ elsewhere. Thus, this system does not have a well-defined relative degree around $x = 0$.

Input-Output Linearization

If a system has a well-defined relative degree then it is input-output linearizable:

$$y^{(r)} = L_f^r h(x) + \underbrace{L_g L_f^{r-1} h(x)}_{\neq 0} u$$

Apply preliminary feedback:

$$\boxed{u = \frac{1}{L_g L_f^{r-1} h(x)} \left(-L_f^r h(x) + v \right)} \tag{4}$$

where v is a new input to be designed. Then, $y^{(r)} = v$ is a linear system in the form of an integrator chain:

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 \\ \dot{\zeta}_2 &= \zeta_3 \\ &\vdots \\ \dot{\zeta}_r &= v \end{aligned}$$

where $\zeta_1 =: y = h(x)$, $\zeta_2 =: \dot{y} = L_f h(x), \ldots, \zeta_r =: y^{(r-1)} = L_f^{r-1} h(x)$. To ensure $y(t) \to 0$ as $t \to \infty$, apply the feedback:

$$\boxed{\begin{aligned} v &= -k_1\zeta_1 - k_2\zeta_2 - \cdots - k_r\zeta_r \\ &= -k_1 h(x) - k_2 L_f h(x) - \cdots - k_r L_f^{r-1} h(x) \end{aligned}} \tag{5}$$

where $k_1, \ldots, k_r$ are such that $s^r + k_r s^{r-1} + \cdots + k_2 s + k_1$ has all roots in the open left half-plane.

Does the controller (4)-(5) achieve asymptotic stability of $x = 0$?

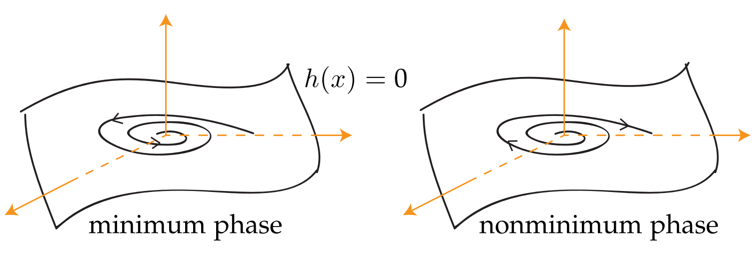
Not necessarily! It renders the $(n-r)$-dimensional manifold:

$$h(x) = L_f h(x) = \cdots = L_f^{r-1} h(x) = 0$$

invariant and attractive. The dynamics restricted to this manifold are called zero dynamics and determine whether or not $x = 0$ is stable.

If the origin of the zero dynamics is asymptotically stable, the system is called minimum phase. If unstable, it is called nonminimum phase.

Example: $n = 3, \quad r = 1$



Finding the Zero Dynamics

Set $y = \dot{y} = \cdots = y^{(r-1)} = 0$ and substitute (4) with $v = 0$, that is:

$$u^* = \frac{-L_f^r h(x)}{L_g L_f^{r-1} h(x)}.$$

The remaining dynamical equations describe the zero dynamics.

Example:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \alpha x_3 + u \\ \dot{x}_3 &= \beta x_3 - u \\ y &= x_1\end{aligned} \tag{6}$$

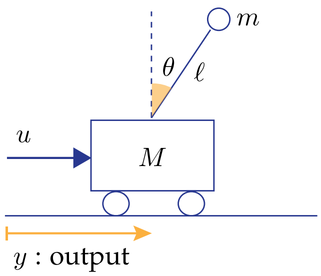
This system has relative degree 2. With $x_1 = x_2 = 0$ and $u^* = -\alpha x_3$, the remaining dynamical equation is

$$\dot{x}_3 = (\alpha + \beta) x_3.$$

Thus this system is minimum phase if $\alpha + \beta < 0$.

Example: Cart/Pole [1]

[1] Simulation code for this example is available online



$$\begin{aligned}\ddot{y} &= \frac{1}{\frac{M}{m} + \sin^2\theta}\left(\frac{u}{m} + \dot{\theta}^2 \ell \sin\theta - g \sin\theta \cos\theta\right) \\ \ddot{\theta} &= \frac{1}{\ell(\frac{M}{m} + \sin^2\theta)}\left(-\frac{u}{m}\cos\theta - \dot{\theta}^2 \ell \cos\theta \sin\theta + \frac{M+m}{m} g \sin\theta\right)\end{aligned} \tag{7}$$

Relative degree = 2.

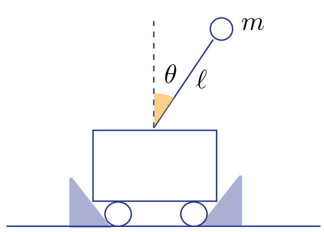
To find the zero dynamics, substitute $y = \dot{y} = 0$, and

$$u^* = -m(\dot{\theta}^2 \ell \sin\theta - g \sin\theta \cos\theta)$$

in the $\ddot{\theta}$ equation:

$$\boxed{\ddot{\theta} = \frac{g}{\ell}\sin\theta.}$$

Same as the dynamics of the pole when the cart is held still:



Nonminimum phase because $\theta = 0$ is unstable for the zero dynamics.

Appendix: Derivation of Cart-Pole Equations of Motion

We will define our state to be $x = [q, \dot{q}]$ with $q = [p, \theta]$ for p being the horizontal position of the cart and θ being the angle of the pole with respect to the vertical. The equations of motion are derived using the Euler-Lagrange equation.

First, we can compute the kinetic energy of our system in terms of the cart's energy and the pendulum's energy:

The simplification in K_{pend} comes from $\cos^2 + \sin^2 = 1$

$$\begin{aligned}
K_{\text{cart}} &= \frac{1}{2} M \dot{p}^2 \\
K_{\text{pend}} &= \frac{1}{2} m (\dot{x}_p^2 + \dot{y}_p^2) \\
&= \frac{1}{2} m ((\dot{p} + \ell \cos(\theta)\dot{\theta})^2 + (\ell \sin(\theta)\dot{\theta})^2) \\
&= \frac{1}{2} m (\dot{p}^2 + 2\ell \cos(\theta)\dot{p}\dot{\theta} + \ell^2 \cos^2(\theta)\dot{\theta}^2 + \ell^2 \sin^2(\theta)\dot{\theta}^2) \\
&= \frac{1}{2} m (\dot{p}^2 + 2\ell \cos(\theta)\dot{p}\dot{\theta} + \ell^2 \dot{\theta}^2) \\
K_{\text{total}} &= K_{\text{cart}} + K_{\text{pend}} \\
&= \frac{1}{2}(M + m)\dot{p}^2 + m\ell \cos(\theta)\dot{p}\dot{\theta} + \frac{1}{2} m \ell^2 \dot{\theta}^2
\end{aligned}$$

Next, we can compute the potential energy of the two subsystems as:

$$\begin{aligned}
P_{\text{cart}} &= 0 \\
P_{\text{pend}} &= -mg\ell \cos(\theta) \\
P_{\text{total}} &= P_{\text{cart}} + P_{\text{pend}} = -mg\ell \cos(\theta)
\end{aligned}$$

Plugging these into our Euler-Lagrange Equation with $L = K - P$ we get:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = F$$

$$\frac{d}{dt}\left(\begin{bmatrix} \frac{\partial(\frac{1}{2}(M+m)\dot{p}^2 + m\ell\cos(\theta)\dot{p}\dot{\theta} + \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell\cos(\theta))}{\partial \dot{p}} \\ \frac{\partial(\frac{1}{2}(M+m)\dot{p}^2 + m\ell\cos(\theta)\dot{p}\dot{\theta} + \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell\cos(\theta))}{\partial \dot{\theta}} \end{bmatrix}\right) - \begin{bmatrix} \frac{\partial(\frac{1}{2}(M+m)\dot{p}^2 + m\ell\cos(\theta)\dot{p}\dot{\theta} + \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell\cos(\theta))}{\partial p} \\ \frac{\partial(\frac{1}{2}(M+m)\dot{p}^2 + m\ell\cos(\theta)\dot{p}\dot{\theta} + \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell\cos(\theta))}{\partial \theta} \end{bmatrix} = \begin{bmatrix} u \\ 0 \end{bmatrix}$$

$$\frac{d}{dt}\left(\begin{bmatrix} (M+m)\dot{p} + m\ell\cos(\theta)\dot{\theta} \\ m\ell\cos(\theta)\dot{p} + m\ell^2\dot{\theta} \end{bmatrix}\right) - \begin{bmatrix} 0 \\ -m\ell\sin(\theta)\dot{p}\dot{\theta} - mg\ell\sin(\theta) \end{bmatrix} = \begin{bmatrix} u \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} (M+m)\ddot{p} - m\ell\sin(\theta)\dot{\theta}^2 + m\ell\cos(\theta)\ddot{\theta} \\ -m\ell\sin(\theta)\dot{\theta}\dot{p} + m\ell\cos(\theta)\ddot{p} + m\ell^2\ddot{\theta} \end{bmatrix} - \begin{bmatrix} 0 \\ -m\ell\sin(\theta)\dot{p}\dot{\theta} - mg\ell\sin(\theta) \end{bmatrix} = \begin{bmatrix} u \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} (M+m)\ddot{p} - m\ell\sin(\theta)\dot{\theta}^2 + m\ell\cos(\theta)\ddot{\theta} \\ m\ell\cos(\theta)\ddot{p} + m\ell^2\ddot{\theta} + mg\ell\sin(\theta) \end{bmatrix} = \begin{bmatrix} u \\ 0 \end{bmatrix}$$

This can be rearranged to separate $\ddot{p}$ and $\ddot{\theta}$:

$$\underbrace{\begin{bmatrix} M+m & m\ell\cos(\theta) \\ m\ell\cos(\theta) & m\ell^2 \end{bmatrix}}_{\boldsymbol{M}} \begin{bmatrix} \ddot{p} \\ \ddot{\theta} \end{bmatrix} + \underbrace{\begin{bmatrix} -m\ell\sin(\theta)\dot{\theta}^2 \\ mg\ell\sin(\theta) \end{bmatrix}}_{\boldsymbol{H}} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\boldsymbol{B}} u$$

Note that this actually follows our standard robotic equations of motion ($M(q)\ddot{q} + H(q,\dot{q}) = Bu$) so we can follow the standard procedure:

$$\ddot{q} = M^{-1}(-H(q,\dot{q}) + Bu)$$

Plugging all of this into a symbolic solver, we arrive at our prevous equations of motion:

$$\ddot{p} = \frac{1}{\frac{M}{m} + \sin^2\theta}\left(\frac{u}{m} + \dot{\theta}^2\ell\sin\theta - g\sin\theta\cos\theta\right)$$

$$\ddot{\theta} = \frac{1}{\ell(\frac{M}{m} + \sin^2\theta)}\left(-\frac{u}{m}\cos\theta - \dot{\theta}^2\ell\cos\theta\sin\theta + \frac{M+m}{m}g\sin\theta\right)$$