ME 6402 – Lecture 14

MIDTERM REVIEW

February 20 2025

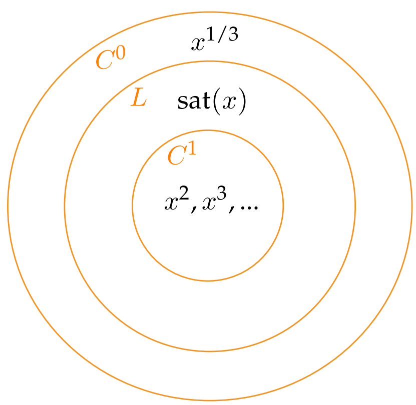
Overview:

- Second-Order Systems (Khalil Chapter 2)
- Mathematical Foundations (Khalil Chapter 3)
- Lyapunov Stability (Khalil Chapter 4)
- Center Manifold Theory (Khalil Chapter 8.1)
- Region of Attraction (Khalil Chapter 8.2)

Mathematical Foundations (Khalil Chapter 3)

It is necessary to determine that solutions to a nonlinear system will exist and be unique when studying their behavior. We can reason about existence and uniqueness using continuity and differentiability of the vector field $f(x)$.

This discussion can be summarized by the following diagram:



Here, C^0 denotes the set of continuous functions, C^1 denotes the set of continuously differentiable functions, and L denotes the set of locally Lipschitz functions, defined as:

Definition: *Locally Lipschitz.* $f(\cdot)$ is locally Lipschitz if every point x^0 has a neighborhood where Lipschitz continuity ($\|f(x) - f(y)\| \leq L\|x - y\|$) holds for all x, y in this neighborhood for some L.

Definition: *Globally Lipschitz.* $f(\cdot)$ is globally Lipschitz if Lipschitz continuity holds for all x, y in the entire domain ($\mathbb{R}^n$). (This is the same as stating that the same L works everywhere.)

The summary of how these function properties relate to existence and uniqueness of solutions is as follows:

1. $f(\cdot)$ is C^0: Solutions exist on a finite interval $[0, T)$
2. $f(\cdot)$ is locally Lipschitz: Solutions exist and are unique on a finite interval $[0, T)$
3. $f(\cdot)$ is globally Lipschitz: Solutions exist and are unique for all $t \geq 0$

Note 1: Since C^1 implies local Lipschitz continuity, we can conclude that C^1 functions exist and have unique solutions on a finite interval.

Note 2: A C^1 function is globally Lipschitz iff ($\frac{\partial f}{\partial x}$) its gradient is bounded.

Second-Order Systems (Khalil Chapter 2)

Essentially Nonlinear Phenomena

- Finite Escape Time
- Multiple Isolated Equilibria
- Limit Cycles
- Chaos

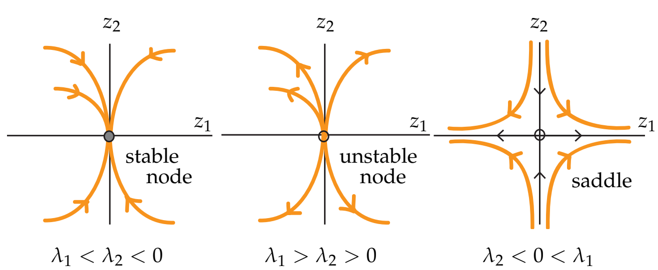
Planar (Second-Order) Dynamical Systems

Solution trajectories ($x(t) = (x_1(t), x_2(t))$) can be represented as curves in the phase plane with $f(x)$ represented as a vector field in the plane. The family of all solution curves is called the *phase portrait*.

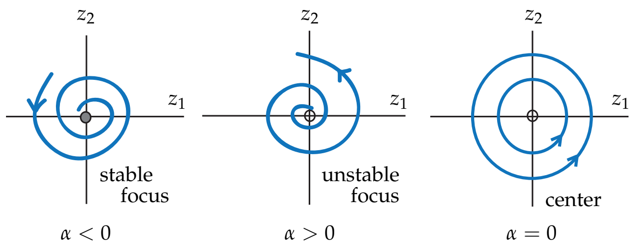
Phase portraits of linear systems ($\dot{x} = Ax$) can be categorized based on the form of their eigenvalues of A:

Note that these diagrams are given in terms of $z = T^{-1}x$ where $J = T^{-1}AT$ is the Jordan form of A. But the same patterns apply (albeit slightly skewed) for the original x coordinates.

1. Distinct Real Eigenvalues: $\lambda_{1,2}$



2. Complex Eigenvalues: $\lambda_{1,2} = \alpha \pm j\beta$



The Hartman-Grobman Theorem allows us to make conclusion about when the phase portraits of the linearized system are qualitatively similar to the nonlinear system.

Theorem: Hartman-Grobman Theorem. *If x^* is a hyperbolic equilibrium of $\dot{x} = f(x)$, then there exists a homeomorphism $z = h(x)$ defined in a neighborhood of x^* such that maps trajectories of $\dot{x} = f(x)$ to those of $\dot{z} = Az$.*

Definition: *Hyperbolic Equilibrium.*: Linearization has no eigenvalues on the imaginary axis. Alternatively, the eigenvalues of A have non-zero real parts.

Scenarios where x^* is not hyperbolic include those with periodic orbits. Thus, we have different tools to reason about when periodic orbits exist or do not exist.

Theorem: Bendixson's Theorem. *For a time-invariant planar system, if $\nabla \cdot f(x) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ is not identically zero and does not change sign in a simply connected region D, then D contains no periodic orbits.*

Alternatively, to reason about when a periodic orbit *must* exist, we use the Poincaré-Bendixson Theorem.

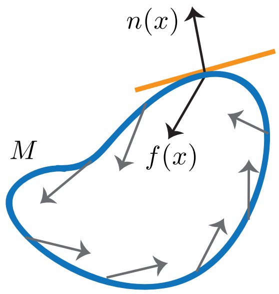
Theorem: Poincaré-Bendixson Theorem. *Suppose M is compact and positively invariant for a planar, time-invariant system. If M contains no equilibrium points, then M contains a periodic orbit.*

This relies on our definition of invariance:

Definition: *Invariance.* A set $M \subset \mathbb{R}^n$ is positively (negatively) invariant if, for each $x_0 \in M$, $\phi(t, x_0) \in M$ for all $t \geq 0$ ($t \leq 0$).

$\phi(t, x_0)$ denotes a trajectory of $\dot{x} = f(x)$ with initial condition $x(0) = x_0$.

Condition for positive invariance: If $f(x) \cdot n(x) \leq 0$ on the boundary, then M is positively invariant.



The condition for "no equilibrium points" can be relaxed to contain one equilibrium as long as its either an unstable focus or an unstable node.

Lastly, we can reason about periodic orbits using index theory. Specifically, since the index of a closed curve is equal to the sum of indices of the equilibria inside, and since a periodic orbit must have an index of $+1$, we can conclude that inside any periodic orbit there must be at least one equilibrium and the indices of the equilibria enclosed must add up to $+1$.

Bifurcations

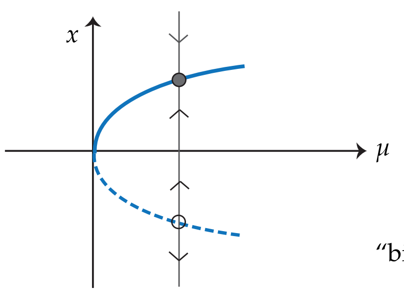
Another tool that is used to study planar systems is bifurcation analysis. Note that birfucations can exist in higher-order systems too, but they are restricted to a one-dimensional manifold. A bifurcation is an abrupt qualitative changes in the phase portrait as a parameter is varied. We introduced a few common examples of bifurcations:

- Fold Bifurcation

  Example: $\dot{x} = \mu - x^2$

  If $\mu > 0$, two equilibria: $x = \mp\sqrt{\mu}$. If $\mu < 0$, no equilibria.

Birfucation diagrams sketch the amplitude of the equilibrium points as a function of the bifurcation parameter. Solid lines represent stable nodes/foci/limit cycles. Dashed lines represent unstable nodes/foci/limit cycles.



"bifurcation diagram"

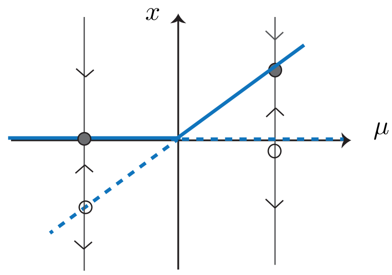
- transcritical bifurcation

  Example: $\dot{x} = \mu x - x^2$

Equilibria: $x = 0$ and $x = \mu$. $\dfrac{\partial f}{\partial x} = \mu - 2x = \begin{cases} \mu & \text{if } x = 0 \\ -\mu & \text{if } x = \mu \end{cases}$

$\mu < 0:$ $x = 0$ is stable, $x = \mu$ is unstable

$\mu > 0:$ $x = 0$ is unstable, $x = \mu$ is stable

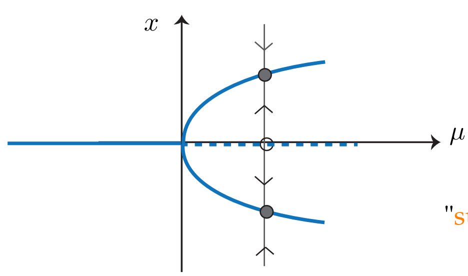


- supercritical pitchfork bifurcation

  Example: $\dot{x} = \mu x - x^3$

  Equilibria: $x = 0$ for all μ, $x = \mp\sqrt{\mu}$ if $\mu > 0$.

	$\mu < 0$	$\mu > 0$
$\left.\frac{\partial f}{\partial x}\right\|_{x=0} = \mu$	stable	unstable
$\left.\frac{\partial f}{\partial x}\right\|_{x=\mp\sqrt{\mu}} = -2\mu$	N/A	stable



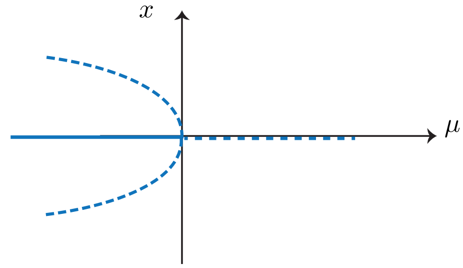
"supercritical pitchfork”

- subcritical pitchfork bifurcation

  Example: $\dot{x} = \mu x + x^3$

  Equilibria: $x = 0$ for all μ, $x = \mp\sqrt{-\mu}$ if $\mu < 0$.

	$\mu < 0$	$\mu > 0$
$\left.\frac{\partial f}{\partial x}\right\|_{x=0} = \mu$	stable	unstable
$\left.\frac{\partial f}{\partial x}\right\|_{x=\mp\sqrt{-\mu}} = -2\mu$	unstable	N/A



"subcritical pitchfork"

Lyapunov Stability (Chapter 4)

Note: We will always assume that the equilibrium is at $x = 0$. This can be achieved using a change of coordinates $\tilde{x} = x - x^*$.

Time-Invariant Systems

Definition: *Stability*. An equilibrium $x = 0$ is stable if for each $\epsilon > 0$, there exist $\delta > 0$ such that

$$\|x(0)\| \leq \delta \Rightarrow \|x(t)\| \leq \epsilon \ \forall t \geq 0$$

Definition: *Unstable*. An equilibrium $x = 0$ is unstable if it is not stable.

Definition: *Asymptotically Stable*. An equilibrium $x = 0$ is asymptotically stable if it is stable and $x \to 0$ for all $x(0)$ in a neighborhood of $x(0)$.

Definition: *Globally Asymptotically Stable*. An equilibrium $x = 0$ is globally asymptotically stable if it is asymptotically stable for all $x(0) \in \mathbb{R}^n$.

Theorem: Lyapunov's Stability Theorem. *If there exists a C^1 function $V : D \to \mathbb{R}$ such that*

$$V(0) = 0, \quad \textit{and} \quad V(x) > 0 \ \forall x \in D - \{0\}$$

and

$$\dot{V}(x) := \frac{\partial V}{\partial x}\dot{x} \leq 0, \ \forall x \in D$$

then $x = 0$ is stable.

Theorem: Asymptotically Stable in the Sense of Lyapunov. *If the condition in the previous theorem is satisfied for $\dot{V} < 0$ for all $x \in D - \{0\}$, then $x = 0$ is asymptotically stable.*

Theorem: Globally Asymptotically Stable in the Sense of Lyapunov. *If the condition in the previous theorem is satisfied for $\dot{V} < 0$ for all $x \in \mathbb{R}^n - \{0\}$, and $\|x\| \to \infty$ implies that $V(x) \to \infty$ (i.e., $V(x)$ is radially unbounded), then $x = 0$ is globally asymptotically stable.*

Theorem: LaSalle-Krasovskii Invariance Principle. *Assume that we have a Lyapunov function with $\dot{V} \leq 0$ for all $x \in D$. Let $S = \{x \in D \text{ s.t. } \dot{V}(x) = 0\}$. If no no solution can stay identically in S other than the trivial solution $x(t) \equiv 0$, then $x = 0$ is asymptotically stable.*

Note: This follows from LaSalle's Invariance Principle, which states that every solution starting in the set S will approach the largest invariant set contained in S. Thus, if this invariant set contains only the origin, then $x(t) \to 0$.

Theorem: Lyapunov for Linear Systems. *A is Hurwitz ($\Re(\lambda) < 0$ for all eigenvalues of A) if and only if for any $Q = Q^T > 0$ there exists $P = P^T > 0$ such that:*

$$A^T P + PA = -Q$$

Moreover, the solution P is unique.

- This theorem originates from the Lyapunov function $V(x) = x^T Px$ with the derivative condition:

$$\dot{V} = x^T(A^T P + PA)x = -x^T Qx < 0$$

- The positive definiteness requirement on Q can be relaxed to

$$A^T P + PA = -Q \leq 0$$

  if (C, A) is observable for $Q = C^T C$.

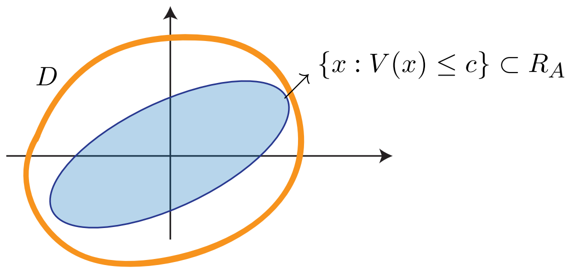
Theorem: Lyapunov's Indirect Method. *Given a linearization*

$$A = \left. \frac{\partial f(x)}{\partial x} \right|_{x=0}$$

Then,

1. *$x = 0$ is asymptotically stable for the nonlinear system if $\Re(\lambda) < 0$ for all eigenvalues of A*
2. *$x = 0$ is unstable for the nonlinear system if $\Re(\lambda) > 0$ for some eigenvalues of A*
3. *The linearization fails to provide information about the stability of the equilibrium if $\Re(\lambda) = 0$ for any eigenvalue of A.*

The region in which the linearization is valid can be estimated using the *Region of Attraction* which "quantifies" local asymptotic stability. It can be estimated by finding the largest level set of V that fits into the set $D = \{x \text{ s.t. } \dot{V}(x) < 0\}$. A simple but conservative choice for the Lyapunov function here is $V(x) = x^T Px$.



Theorem: Alternative Statement of Lyapunov's Theorem. *If the following conditions are satisfied:*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$
$$\dot{V}(x) \leq -\alpha_3(\|x\|)$$

for α_i being a class-$\mathcal{K}$ function, then $x = 0$ is asymptotically stable. Moreover,

$$\|x(t)\| \leq \alpha_1^{-1}\left(\beta(\alpha_2(\|x(0)\|), t - t_0)\right), \; \forall t \geq t_0$$

where $\beta \in \mathcal{KL}$ is the solution to the IVP

$$\dot{y} = -\alpha_3(\alpha_2^{-1}(y)), \; y(t_0) = V(x(t_0))$$

Time-Varying Systems

For time-varying systems $\dot{x} = f(t, x)$ we have different definitions for stability.

Definition: *Stability.* $x = 0$ is stable if for every $\epsilon > 0$ and t_0, there exists $\delta > 0$ such that

$$\|x(t_0)\| \leq \delta(t_0, \epsilon) \Rightarrow \|x(t)\| \leq \epsilon, \; \forall t \geq t_0$$

Definition: *Uniform Stability.* If the same δ works for all t_0 (i.e., $\delta = \delta(\epsilon)$), then $x = 0$ is uniformly stable. Alternatively, $x = 0$ is uniformly stable if there exists a class-$\mathcal{K}$ function $\alpha(\cdot)$ and a constant $c > 0$ such that:

$$\|x(t)\| \leq \alpha(\|x(t_0)\|)$$

for all $t \geq t_0$ and for every initial condition such that $\|x(t_0)\| \leq c$

Definition: *Uniformly Asymptotically Stable.* $x = 0$ is uniformly asymptotically stable if there exists a class $\mathcal{KL}$ function $\beta(\cdot, \cdot)$ such that:

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0)$$

for all $t \geq t_0$ and for every initial condition such that $\|x(t_0)\| \leq c$.

Definition: *Globally Uniformly Asymptotically Stable.* $x = 0$ is globally uniformly asymptotically stable if it is uniformly asymptotically stable for all $x(0) \in \mathbb{R}^n$.

Definition: *Uniformly Exponentially Stable.* $x = 0$ is uniformly exponentially stable if $\beta(r, s) = kre^{-\lambda s}$ for some $k, \lambda > 0$ such that

$$\|x(t)\| \leq k\|x(t_0)\|e^{-\lambda(t-t_0)}$$

for all $t \geq t_0$ and for every initial condition such that $\|x(t_0)\| \leq c$.

The Theorems for Lyapunov's Stability applied to time-varying systems can be summarized as follows:

- If $W_1(x) \leq V(t, x) \leq W_2(x)$ and $\dot{V}(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq 0$ for some positive definite functions $W_1(\cdot)$ and $W_2(\cdot)$ on a domain D that includes the origin, then $x = 0$ is uniformly stable.
- If, further, $\dot{V}(t, x) \leq -W_3(x)$ for all $x \in D$ for some positive definite $W_3(\cdot)$, then $x = 0$ is uniformly asymptotically stable.
- If $D = \mathbb{R}^n$ and $W_1(\cdot)$ is radially unbounded, then $x = 0$ is globally uniformly asymptotically stable.
- If $W_i = k_i\|x\|^a$ for $i = 1, 2, 3$ and some constants $k_1, k_2, k_3, a > 0$, then $x = 0$ is uniformly exponentially stable.
- If W_3 is positive semidefinite. Then if $W_1(\cdot)$ is radially unbounded, $f(t, x)$ is locally Lipschitz in x and bounded in t, and $W_3(\cdot)$ is C^1, then $W_3(x(t)) \to 0$ as $t \to \infty$. This proves convergence to $S = \{x \text{ s.t. } W_3(x) = 0\}$.

Lastly, for a linear time-varying system $\dot{x} = A(t)x$, $V(t, x) = x^T P(t)x$ proves uniform exponential stability (equivalent to uniform asymptotic stability for a linear time-varying system) if

1. $\dot{P}(t) + A^T(t)P(t) + P(t)A(t) = -Q(t)$
2. for some bounded $Q(t)$: $0 < k_3 I \leq Q(t)$ for all t
3. and if $P(t)$ remains bounded $P(t)$: $0 < k_1 I \leq P(t) \leq k_2 I$

The converse is also true (given a time-varying linear system with a uniformly exponentially stable originthere exists a symmetric $P(t)$ satisfying the conditions above).

Center Manifold Theory (Khalil 8.1)

If the linearization $A = \frac{\partial f}{\partial x}(x)_{x=0}$ has some eigenvalues with zero real part, and the rest have negative real parts, our linearization still fails to capture the dynamics of our system. However, we can use the Center Manifold Theorem to find a lower-dimensional manifold that captures the dynamics of the system near the equilibrium.

The center manifold theory begins by transforming our system into the form:

$$\begin{bmatrix} y \\ z \end{bmatrix} = Tx$$

such that

$$TAT^{-1} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

with the eigenvalues of A_1 having zero real parts and the eigenvalues of A_2 having negative real parts. We will suppose that $y \in \mathbb{R}^k$ (A had k eigenvalues with zero real parts) and $z \in \mathbb{R}^{n-k}$ (A had $n-k$ eigenvalues with negative real parts).

Theorem: Center Manifold Theorem. *There exists an invariant manifold $z = h(y)$ defined in a neighborhood of $x = 0$ such that*

$$h(0) = 0, \quad \frac{\partial h}{\partial y}(0) = 0$$

$z = h(y)$ is called a center manifold and results in the reduced system

$$\dot{y} = A_1 y + g_1(y, h(y)), \quad y \in \mathbb{R}^k$$

If $y = 0$ is asymptotically stable (resp. unstable) for the reduced system, then $x = 0$ is asymptotically stable (resp. unstable) for the full system $\dot{x} = f(x)$.

We can solve for the center manifold by defining $w = z - h(y)$ and differentiating $\dot{w} = \dot{z} - \frac{\partial h}{\partial y}\dot{y}$. We can then solve for $h(y)$ by enforcing our invariance condition $\dot{w} = 0$.

In scenarios where $y \in \mathbb{R}$ (scalar y), we can expand $h(y)$ as:

$$h(y) = h_2 y^2 + h_3 y^3 + \cdots$$

and solve for the coefficients. (Note that $h_0 = h_1 = 0$ since $h(0) = \frac{\partial h}{\partial y}(0) = 0$).