ME 6402 – Lecture 13 [1]

BACKSTEPPING

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Overview:

- Lyapunov-based design: Backstepping

Additional Reading:

- Khalil, Chapter 14.3
- Sastry, Chapter 6.8

Clarification

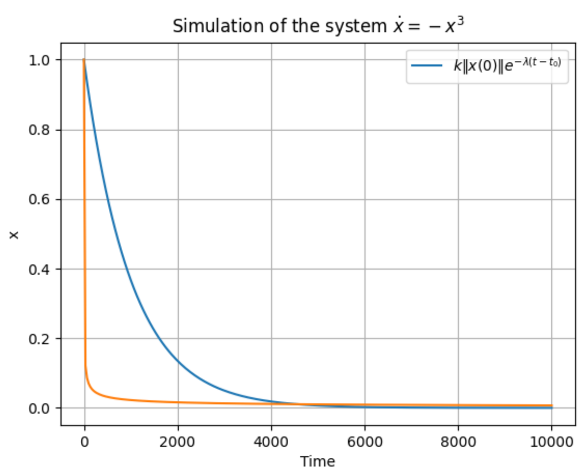
The condition for uniform exponential stability in the Lyapunov theorem is: If $W_i(x) = k_i\|x\|^a$, $i = 1, 2, 3$, for some constants $k_1, k_2, k_3, a > 0$, then $x = 0$ is uniformly exponentially stable. Notice here that a must be constant across all three terms.

Recall the example:

$$\dot{x} = -g(t)x^3 \quad \text{where} \quad g(t) \geq 1 \quad \text{for all} \quad t$$

$$V(x) = \frac{1}{2}x^2 \quad \Rightarrow \quad \dot{V}(t,x) = -g(t)x^4 \leq -x^4 \triangleq W_3(x)$$

This is *not* exponential stability since $W_1(x) = W_2(x) = \frac{1}{2}\|x\|^2$, but $W_3(x) = \|x\|^4$. Thus, since $2 \neq 4$ (the exponents) the system is not uniformly exponentially stable. This can be confirmed by plotting the system evolution for $x(0) = 1$ and $g(t) = 1$. We cannot bound the solution by $\|x(t)\| \leq k\|x(t_0)\|e^{-\lambda(t-t0)}$.



Model Reference Adaptive Control (MRAC) – Revisited

Consider the first order system

$$\dot{y} = a^* y + u$$

where $a^* \in \mathbb{R}$ is unknown.

Reference model:

$$\dot{y}_m = -a y_m + r(t) \quad a > 0,\ r(t) : \text{reference signal.}$$

The reference signal is the desired input or trajectory that the controlled system should follow.

<u>Goal:</u> Design a controller that guarantees $y(t) - y_m(t) \to 0$ without the knowledge of a^*.

We start by defining a new error variable $e(t) = y(t) - y_m(t)$ and differentiate to get:

$$\begin{aligned}\dot{e} &= \dot{y} - \dot{y}_m = a^* y + u - (-a y_m + r(t)) \\ &= a^* y + a y_m + u - r(t).\end{aligned}$$

If we were to select $u = -\underbrace{(a^* + a)}_{k^*} y + r(t)$, this simplifies to:

$$\dot{e} = -a(y - y_m) = -ae \quad \Rightarrow \quad e(t) \to 0 \text{ exponentially.}$$

However, since we do not know a^*, then k^* is unknown. To address this, we can design an adaptive controller:

$$u = -k(t) y + r(t)$$

where $\dot{k}(t)$ is to be designed. Plugging this controller into our error dynamics, we get:

$$\begin{aligned}\dot{e} &= \dot{y} - \dot{y}_m \\ &= a^* y + (-k(t) y + r(t)) - (-a y_m + r(t)) \\ &= a^* y + a y_m - k(t) y + a y - a y \\ &= \underbrace{(a^* + a)}_{k^*} y - a \underbrace{(y - y_m)}_{e} - k(t) y \\ &= -ae - \underbrace{(k(t) - k^*)}_{=: \tilde{k}(t)} y\end{aligned}$$

Use the Lyapunov function: $V = \frac{1}{2}e^2 + \frac{1}{2}\tilde{k}^2$:

$$\begin{aligned}\dot{V} &= e\dot{e} + \tilde{k}\dot{\tilde{k}} \\ &= e(-ae - \tilde{k}y) + \tilde{k}\dot{\tilde{k}} \\ &= -ae^2 - \tilde{k}ey + \tilde{k}\dot{\tilde{k}} \\ &= -ae^2 + \tilde{k}(\dot{\tilde{k}} - ey).\end{aligned}$$

Note $\dot{\tilde{k}} = \dot{k}$ (since k^* is constant) and choose $\boxed{\dot{k} = ey}$ so that $\dot{V} = -ae^2$.

This guarantees stability of $(e, \tilde{k}) = (0, 0)$ and boundedness of $(e(t), \tilde{k}(t))$ since the level sets of $V = \frac{1}{2}e^2 + \frac{1}{2}\tilde{k}^2$ are positively invariant. To conclude that $e(t) \to 0$, it remains to show that $f(t, x)$ is bounded in t. This can be shown by assuming that $r(t)$ is bounded. Thus, we can conclude from $\dot{V} = -ae^2$ that $e(t) \to 0$.

Backstepping

Khalil (Sec. 14.3), Sastry (Sec. 6.8)

Feedback stabilization: Given a control-affine nonlinear system[2]

$$\dot{x} = f(x) + g(x)u \tag{1}$$

with input $u \in \mathbb{R}$ and smooth functions $f : D \to \mathbb{R}^n$ and $g : D \to \mathbb{R}^n$, design a control law $u = k(x)$ such that $x = 0$ is asymptotically stable for the closed-loop system:

$$\dot{x} = f(x) + g(x)k(x).$$

[2] We will show in a later lecture that all mechanical (robotic) systems can be cast in this form.

Backstepping is a technique that simplifies this task for a class of systems.

Suppose a stabilizing feedback $u = k(\eta)$, $k(0) = 0$, is available for:

$$\dot{\eta} = F(\eta) + G(\eta)u \quad \eta \in \mathbb{R}^n, u \in \mathbb{R}, \ F(0) = 0,$$

along with a Lyapunov function V such that

$$\frac{\partial V}{\partial \eta}\Big(F(\eta) + G(\eta)k(\eta)\Big) \leq -W(\eta) < 0 \quad \forall \eta \neq 0.$$

Can we modify $k(\eta)$ to stabilize the augmented system below?

$$\boxed{\begin{aligned} \dot{\eta} &= F(\eta) + G(\eta)\xi \\ \dot{\xi} &= u. \end{aligned}}$$

Define the error variable $\boxed{z = \xi - k(\eta)}$ and change variables: $(\eta, \xi) \to (\eta, z)$:

$$\begin{aligned} \dot{\eta} &= F(\eta) + G(\eta)k(\eta) + G(\eta)z \\ \dot{z} &= u - \dot{k}(\eta, z) \end{aligned}$$

where $\dot{k}(\eta, z) = \frac{\partial k}{\partial \eta}\Big(F(\eta) + G(\eta)k(\eta) + G(\eta)z\Big)$. Take the new Lyapunov function:

$$V_+(\eta, z) = V(\eta) + \frac{1}{2}z^2.$$

$$\begin{aligned}\dot{V}_+ &= \frac{\partial V}{\partial \eta}\dot{\eta} + z\dot{z} \\ &= \underbrace{\frac{\partial V}{\partial \eta}\Big(F(\eta) + G(\eta)k(\eta)\Big)}_{\leq -W(\eta)} + \underbrace{\frac{\partial V}{\partial \eta}G(\eta)z + z(u - \dot{k})}_{= z\left(u - \dot{k} + \frac{\partial V}{\partial \eta}G(\eta)\right)}\end{aligned}$$

Let: $\boxed{u = \dot{k} - \frac{\partial V}{\partial \eta}G(\eta) - Kz, \quad K > 0.}$

Then, $\dot{V}_+ \leq -W(\eta) - Kz^2 \;\Rightarrow\; (\eta, z) = 0$ is asymptotically stable.

- Above we discussed backstepping over a pure integrator. The main idea generalizes trivially to:

$$\boxed{\begin{aligned}\dot{\eta} &= F(\eta) + G(\eta)x \\ \dot{x} &= f(\eta, x) + g(\eta, x)u\end{aligned}}$$

where $\eta \in \mathbb{R}^n$, $x \in \mathbb{R}$, and $g(\eta, x) \neq 0$ for all $(\eta, x) \in \mathbb{R}^{n+1}$.

With the preliminary feedback

$$u = \frac{1}{g(\eta, x)}(-f(\eta, x) + v) \tag{2}$$

the x-subsystem becomes a pure integrator: $\dot{x} = v$. Substituting the backstepping control law from above:

$$v = \dot{k} - \frac{\partial V}{\partial \eta}G(\eta) - Kz, \quad z \triangleq x - k(\eta), \quad K > 0$$

into (2), we get:

$$\boxed{u = \frac{1}{g(\eta, x)}\left(-f(\eta, x) + \dot{k} - \frac{\partial V}{\partial \eta}G(\eta) - Kz\right).}$$

- Backstepping can be applied recursively to systems of the form:[3]

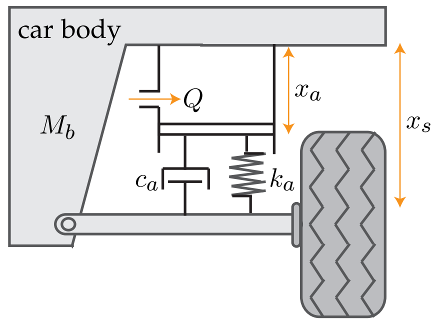
$$\begin{aligned}\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)x_3 \\ \dot{x}_3 &= f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)x_4 \\ &\;\vdots \\ \dot{x}_n &= f_n(x) + g_n(x)u\end{aligned} \tag{3}$$

where $g_i(x_1, \ldots, x_i) \neq 0$ for all $x \in \mathbb{R}^n$, $i = 2, 3, \cdots, n$.

[3] Systems of this form are called "strict feedback systems."

Design example: Active suspension

Krstić et al., Nonlinear and Adaptive Control Design, Section 2.2.2.



$$M_b\ddot{x}_s = -k_a(x_s - x_a) - c_a(\dot{x}_s - \dot{x}_a)$$

$$\dot{x}_a = \frac{1}{A}Q \qquad A\text{: effective piston surface}$$

$$\text{Flow: } \dot{Q} = -c_f Q + k_f u \qquad u\text{: current applied to the solenoid valve (control input)}$$

c_f: outflow or dissipation rate

k_f: flow input control gain

Define state variables: $x_1 = x_s$, $x_2 = \dot{x}_s$, $x_3 = x_a$, $x_4 = Q$:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{k_a}{M_b}(x_1 - x_3) - \frac{c_a}{M_b}(x_2 - \frac{1}{A}x_4) \\
\dot{x}_3 &= \frac{1}{A}x_4 \\
\dot{x}_4 &= -c_f x_4 + k_f u.
\end{aligned} \tag{4}$$

This system is not in strict recursive form due to the x_4 term in $\dot{x}_2$. To overcome this problem define:

$$\begin{aligned}
\bar{x}_3 &\triangleq \frac{k_a}{M_b}x_3 + \frac{c_a}{M_b A}x_4 \\
\xi &\triangleq x_3
\end{aligned}$$

and change variables to $(x_1, x_2, \bar{x}_3, \xi)$:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{k_a}{M_b}x_1 - \frac{c_a}{M_b}x_2 + \bar{x}_3 \\
\dot{\bar{x}}_3 &= \frac{k_a - c_a c_f}{M_b A}x_4 + \frac{c_a k_f}{M_b A}u.
\end{aligned}$$

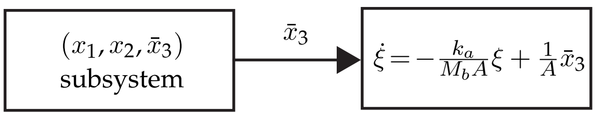
Two steps of backstepping starting with the virtual control law:

$$x_2 = k(x_1) = -c_1 x_1 - k_1 x_1^3$$

[4] will stabilize the $(x_1, x_2, \bar{x}_3)$ subsystem. Full $(x_1, x_2, \bar{x}_3, \xi)$ system:

The stiff nonlinearity $k_1 x_1^3$ prevents large excursions of x_1.

[4] $u = \frac{1}{g(\eta,x)}\left(-f(\eta,x) + \dot{k} - \frac{\partial V}{\partial \eta}G(\eta) - Kz\right)$



The ξ-subsystem is an asymptotically stable linear system driven by $\bar{x}_3$; therefore the full system is stabilized.

Other Examples

Example:

$$\begin{aligned} \dot{x}_1 &= x_1^2 + x_2 \\ \dot{x}_2 &= u \end{aligned}$$

Treat x_2 as "virtual" control input for the x_1-subsystem:

$$\begin{aligned} k(x_1) &= -Kx_1 - x_1^2 \quad K > 0 \\ V_1(x_1) &= \frac{1}{2}x_1^2. \end{aligned}$$

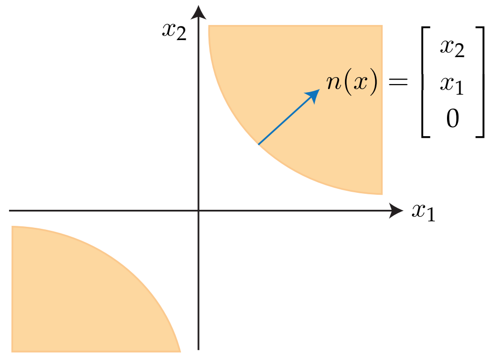
Apply backstepping:

$$\begin{aligned} z_2 &= x_2 - k(x_1) = x_2 + Kx_1 + x_1^2 \\ \dot{z}_2 &= u - \dot{k} \\ u &= \dot{k} - \frac{\partial V_1}{\partial x_1} - k_2 z_2, \quad k_2 > 0 \\ &= \underbrace{-(K + 2x_1)(x_1^2 + x_2)}_{=\dot{k}} - \underbrace{x_1}_{=\frac{\partial V_1}{\partial x_1}} - k_2 \underbrace{(x_2 + Kx_1 + x_1^2)}_{=z_2}. \end{aligned}$$

Example 2:

$$\begin{aligned} \dot{x}_1 &= (x_1 x_2 - 1)x_1^3 + (x_1 x_2 + x_3^2 - 1)x_1 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u. \end{aligned} \tag{5}$$

Not in strict feedback form because x_3 appears too soon. In fact, this system is not globally stabilizable because the set $x_1 x_2 \geq 2$ is positively invariant regardless of u:



To see this, note that

$$n(x) \cdot f(x,u) = [(x_1x_2 - 1)x_1^3 + (x_1x_2 + x_3^2 - 1)x_1]x_2 + x_3x_1$$

and substitute $x_1x_2 = 2$:

$$\begin{aligned}
&= \left(x_1^3 + (1 + x_3^2)x_1\right)x_2 + x_3x_1 \\
&= \left(x_1^2 + (1 + x_3^2)\right)x_1x_2 + x_3x_1 \\
&= 2x_1^2 + 2(1 + x_3^2) + x_3x_1 \\
&= \underbrace{2x_1^2 + x_3x_1 + 2x_3^2}_{\geq 0} + 2 > 0.
\end{aligned}$$

Example 3:

$$\begin{aligned}
\dot{x}_1 &= x_1^2 x_2 \\
\dot{x}_2 &= u
\end{aligned} \tag{6}$$

Treat x_2 as virtual control and let $\boxed{\alpha_1(x_1) = -x_1}$ which stabilizes the x_1-subsystem, as verified with Lyapunov function $V_1(x_1) = \frac{1}{2}x_1^2$.
Then $z_2 := x_2 - \alpha_1(x_1)$ satisfies $\dot{z}_2 = u - \dot{\alpha}_1$, and

$$u = \dot{\alpha}_1 - \frac{\partial V_1}{\partial x_1}x_1^2 - k_2z_2 = -x_1^2x_2 - x_1^3 - k_2(x_2 + x_1)$$

achieves global asymptotic stability:

$$V = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2 \quad \Rightarrow \quad \dot{V} = -x_1{}^4 - k_2z_2^2.$$

Note that we can't conclude exponential stability due to the quartic term x_1^4 above (recall the Lyapunov sufficient condition for exponential stability in Lecture 11, p.2). In fact, the linearization of the closed-loop system proves the lack of exponential stability:

$$\begin{bmatrix} 0 & 0 \\ 0 & -k_2 \end{bmatrix} \rightarrow \lambda_{1,2} = 0, -k_2.$$