# *ME 6402 – Lecture 12* [1]

## TIME-VARYING SYSTEMS AND LYAPUNOV DESIGN

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Overview:

- Linear Time-Varying Systems
- Differential Lyapunov Equation
- Lyapunov Design Examples

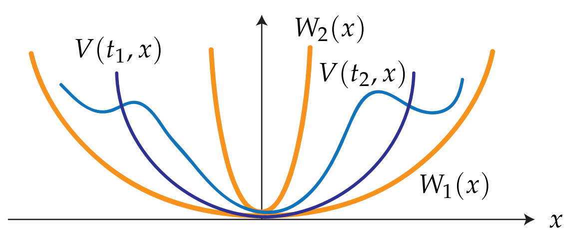
Additional Reading:

- Khalil, Chapter 8.3, 4.6

## *Review of Lyapunov Stabilty Theorem for Time-Varying Systems*

The Lyapunov stability theorems for time-varying systems introduced in the last lecture can be summarized as follows:

1. If $W_1(x) \leq V(t,x) \leq W_2(x)$ and $\dot{V}(t,x) \triangleq \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq 0$ for some positive definite functions $W_1(\cdot)$, $W_2(\cdot)$ on a domain $D$ that includes the origin, then $x = 0$ is uniformly stable.




2. If, further, $\dot{V}(t,x) \leq -W_3(x)$ $\forall x \in D$ for some positive definite $W_3(\cdot)$, then $x = 0$ is uniformly asymptotically stable.

3. If $D = \mathbb{R}^n$ and $W_1(\cdot)$ is radially unbounded, then $x = 0$ is globally uniformly asymptotically stable.

4. If $W_i(x) = k_i|x|^a$, $i = 1,2,3$, for some constants $k_1, k_2, k_3, a > 0$, then $x = 0$ is uniformly exponentially stable.

Example:

$$\dot{x} = -g(t)x^3 \quad \text{where} \quad g(t) \geq 1 \quad \text{for all} \quad t$$

$$V(x) = \frac{1}{2}x^2 \quad \Rightarrow \quad \dot{V}(t,x) = -g(t)x^4 \leq -x^4 \triangleq W_3(x)$$

Globally uniformly asymptotically stable but not exponentially stable.

What if $W_3(\cdot)$ is only semidefinite?

Khalil, Section 8.3

Lasalle-Krasovskii Invariance Principle is not applicable to time-varying systems. Instead, we will have to use the following (weaker) result.

**Theorem:** . *Suppose* $W_1(x) \leq V(t,x) \leq W_2(x)$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W_3(x),$$

*where* $W_1(\cdot), W_2(\cdot)$ *are positive definite and* $W_3(\cdot)$ *is positive semidefinite. Suppose, further,* $W_1(\cdot)$ *is radially unbounded,* $f(t,x)$ *is locally Lipschitz in* $x$ *and bounded in* $t$*, and* $W_3(\cdot)$ *is* $C^1$*. Then*

$$\boxed{W_3(x(t)) \to 0 \text{ as } t \to \infty.}$$

Note: This proves convergence to $S = \{x : W_3(x) = 0\}$ whereas the Invariance Principle, when applicable, guarantees convergence to the largest invariant set within $S$.

## *Linear Time-Varying Systems*

Khalil Section 4.6, Sastry Section 5.7

Our linear time-varying system can be first introduced as simply a special case for our general time-varying system:

$$\dot{x} = A(t)x \qquad x(t) = \Phi(t,t_0)x(t_0) \tag{1}$$

The state transition matrix $\Phi(t,t_0)$ satisfies the equations:

$$\frac{\partial}{\partial t}\Phi(t,t_0) = A(t)\Phi(t,t_0) \tag{2}$$

$$\frac{\partial}{\partial t_0}\Phi(t,t_0) = -\Phi(t,t_0)A(t_0) \tag{3}$$

- No eigenvalue test for stability in the time-varying case:
- For linear systems uniform asymptotic stability is equivalent to uniform exponential stability:

  **Theorem:** (4.11 in Khalil[2]). $x = 0$ *is uniformly asymptotically stable if and only if*

  $$\|\Phi(t,t_0)\| \leq k e^{-\lambda(t-t_0)} \text{ for some } k > 0,\ \lambda > 0.$$

[2] Khalil Thm. 4.11, Sastry Thm. 5.33

Example: $\dot{x} = A(t)x$. Take $V(x) = x^T P(t) x$:

$$\begin{aligned}\dot{V}(x) &= x^T \dot{P}(t)x + \dot{x}^T P(t)x + x^T P(t)\dot{x} \\ &= x^T \underbrace{(\dot{P} + A^T P + PA)}_{\triangleq -Q(t)} x\end{aligned}$$

If $k_1 I \le P(t) \le k_2 I$ and $k_3 I \le Q(t)$, $k_1, k_2, k_3 > 0$, then

$$k_1|x|^2 \le V(t,x) \le k_2|x|^2 \quad \text{and} \quad \dot{V}(t,x) \le -k_3|x|^2$$

$\Rightarrow$ global uniform exponential stability.

- $V(t,x) = x^T P(t) x$ proves uniform exp. stability if

$$\begin{aligned} &\text{(i)} && \dot{P}(t) + A^T(t)P(t) + P(t)A(t) = -Q(t) \\ &\text{(ii)} && 0 < k_1 I \le P(t) \le k_2 I \\ &\text{(iii)} && 0 < k_3 I \le Q(t) \text{ for all } t. \end{aligned}$$

The converse is also true:

Theorem: Suppose $x = 0$ is uniformly exponentially stable, $A(t)$ is continuous and bounded, $Q(t)$ is continuous and symmetric, and there exist $k_3, k_4 > 0$ such that

$$0 < k_3 I \le Q(t) \le k_4 I \;\; \text{for all } t.$$

Then, there exists a symmetric $P(t)$ satisfying (i)–(ii) above.

- For stable linear systems, there always exists quadratic Lyapunov functions
- Find them by choosing any positive definite $Q(t)$ and solve (differential) Lyapunov equation.

Proof:

Time-invariant: $P = \int_0^\infty e^{A^T\tau} Q e^{A\tau} d\tau$

Time-varying: $P(t) = \int_t^\infty \Phi^T(\tau,t) Q(\tau) \Phi(\tau,t) d\tau$

Using the Leibniz rule, property (3), and $\Phi(t,t) = I$ we obtain:

$$\begin{aligned} \dot{P}(t) &= \int_t^\infty \left( \frac{\partial}{\partial t}\Phi^T(\tau,t)Q(\tau)\Phi(\tau,t) + \Phi^T(\tau,t)Q(\tau)\frac{\partial}{\partial t}\Phi(\tau,t) \right) d\tau \\ &\qquad\qquad - \Phi^T(t,t)Q(t)\Phi(t,t) \\ &= \int_t^\infty \left( -A^T(t)\Phi^T(\tau,t)Q(\tau)\Phi(\tau,t) - \Phi^T(\tau,t)Q(\tau)\Phi(\tau,t)A(t) \right) d\tau \\ &\qquad\qquad - \Phi^T(t,t)Q(t)\Phi(t,t) \\ &= -A^T(t)P(t) - P(t)A(t) - Q(t). \end{aligned}$$

## *Lyapunov-based Feedback Design Examples*

We will next discuss a few different ways in which Lyapunov theory is used. This will include adaptive control (this lecture), backstepping (next lecture) and Control Lyapunov Functions (later).

## *Model Reference Adaptive Control (MRAC)*

Let's consider the first order system

$$\dot{y} = a^* y + u$$

where $a^* \in \mathbb{R}$ is unknown.

Goal: Stabilize the origin even when $a^*$ is unknown and design a controller that learns $a^*$. To achieve this *estimator convergence*, we will introduce the notion of a reference model.

Reference model:

$$\dot{y}_m = -a y_m + r(t) \quad a > 0,\ r(t) : \text{reference signal.}$$

The reference signal is the desired input or trajectory that the controlled system should follow.

Goal: Design a controller that guarantees $y(t) - y_m(t) \to 0$ without the knowledge of $a^*$.

If we knew $a^*$, we would choose:

$$u = -\underbrace{(a^* + a)}_{=: k^*} y + r(t) \quad \Rightarrow \quad \dot{y} = -ay + r(t).$$

The tracking error $\boxed{e(t) := y(t) - y_m(t)}$ then satisfies:

$$\dot{e} = -ae \ \Rightarrow \ e(t) \to 0 \text{ exponentially.}$$

*Adaptive* design when $a^*$ (therefore, $k^*$) is unknown:

$$u = -k(t)y + r(t)$$

where $\dot{k}(t)$ is to be designed. Then:

$$\dot{e} = \dot{y} - \dot{y}_m = a^* y - k(t)y + a y_m = -ae - \underbrace{(k(t) - k^*)}_{=: \tilde{k}(t)} y \tag{4}$$

where adding and subtracting $ay$ gives the final equality.

Use the Lyapunov function: $V = \frac{1}{2}e^2 + \frac{1}{2}\tilde{k}^2$:

$$\begin{aligned} \dot{V} &= e\dot{e} + \tilde{k}\dot{\tilde{k}} \\ &= -ae^2 - \tilde{k}ey + \tilde{k}\dot{\tilde{k}} \\ &= -ae^2 + \tilde{k}(\dot{\tilde{k}} - ey). \end{aligned}$$

Note $\dot{\tilde{k}} = \dot{k}$ and choose $\boxed{\dot{k} = ey}$ so that $\dot{V} = -ae^2$.

This guarantees stability of $(e, \tilde{k}) = (0, 0)$ and boundedness of $(e(t), \tilde{k}(t))$ since the level sets of $V = \frac{1}{2}e^2 + \frac{1}{2}\tilde{k}^2$ are positively invariant. In addition, if $r(t)$ is bounded, then $y_m(t)$ in the reference model is bounded, and so is $y(t) = y_m(t) + e(t)$. Then we can apply the Theorem from the start of lecture to the time-varying model

$$\dot{e} = -ae - y(t)\tilde{k}, \quad \dot{\tilde{k}} = y(t)e,$$

and conclude from $\dot{V} = -ae^2$ that $e(t) \to 0$.

Whether $\tilde{k}(t) \to 0$ ($k(t) \to k^*$) depends on further properties of the reference signal $r(\cdot)$ that are beyond the scope of this lecture.