# *Nonlinear Control Systems—Lecture 1 Notes*[1]

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## *Linear Systems*

$$\dot{x} = Ax, \quad x(t_0) = x_0 \in \mathbb{R}^n \tag{1}$$

We use the shorthand notation $\dot{x} = f(x)$ for $\frac{d}{dt}x(t) = f(x(t))$.

Here, $A$ is an $n \times n$ constant matrix. This linear system has the following properties:

1. Solutions always exist, and are given in closed form

$$x(t) = e^{A(t-t_0)}x_0, \ t \geq t_0$$

2. Solutions exist for all $-\infty < t < \infty$
3. Solutions are unique
4. The set of equilibrium points is the nullspace of $A$ (i.e., connected)
5. Periodic solutions are only marginally stable, never stable (asympotically or exponentially)

## *Nonlinear Systems*

In comparison, nonlinear systems are more complex but also more expressive. We will consider nonlinear systems of the form:

$$\dot{x} = f(x), \ x(t_0) \in \mathbb{R}^n \tag{2}$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$.

This system is time-invariant. We can also consider time-varying systems:

| | | |
|---|---|---|
| $\dot{x} = f(x)$ | $f : \mathbb{R}^n \to \mathbb{R}^n$ | time-invariant (autonomous) |
| $\dot{x} = f(t, x)$ | $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ | time-varying (non-autonomous) |

When the system has a control input $u \in \mathbb{R}^m$, the linear and nonlinear system dynamics are:

$$\dot{x} = Ax + Bu \quad \longrightarrow \quad \dot{x} = f(x, u) \tag{3}$$

Sometimes the nonlinear system can be written as $\dot{x} = f(x) + g(x)u$, which is called *control-affine* form.

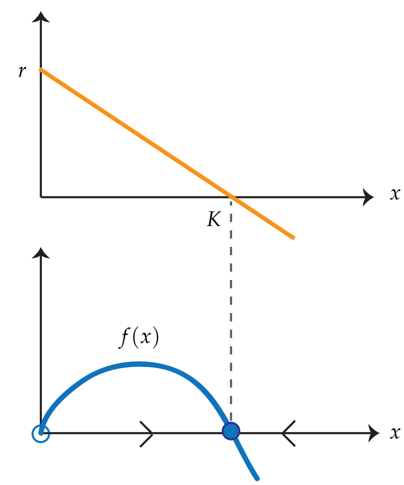
*Nonlinear System Analysis and Design*

- Analysis (first half of course): Determine stability, convergence, etc of $\dot{x} = f(x)$
- Design (second half of course): Choose $u$ as a function of $x$ to achieve desired behavior

*Motivating Scalar Example*

*Logistic growth model* in population dynamics

$$\dot{x} = f(x) = \underbrace{r\left(1 - \frac{x}{K}\right)}_{\text{growth rate}} x, \qquad r > 0, \quad K > 0 \tag{4}$$




$x > 0$ denotes the population, $K$ is called the carrying capacity, and $r$ is the intrinsic growth rate.

For systems with a scalar state variable $x \in \mathbb{R}$, stability can be determined from the sign of $f(x)$ around the equilibrium. In this example $f(x) > 0$ for $x \in (0, K)$, and $f(x) < 0$ for $x > K$; therefore

$$\begin{aligned} x = 0 &\quad \text{unstable equilibrium} \\ x = K &\quad \text{asymptotically stable.} \end{aligned}$$

In general, $x = x^*$ is an equilibrium for $\dot{x} = f(x)$ if $f(x^*) = 0$

*Linearization*

Local stability properties of $x^*$ can be determined by linearizing the vector field $f(x)$ at $x^*$:

Note this is the same as $f(x) \approx f(x^*) + f'(x^*)(x - x^*)$.

$$f(x^* + \tilde{x}) = \underbrace{f(x^*)}_{=\ 0} + \underbrace{\left.\frac{\partial f}{\partial x}\right|_{x=x^*}}_{\triangleq\ A} \tilde{x} + \text{higher order terms} \tag{5}$$

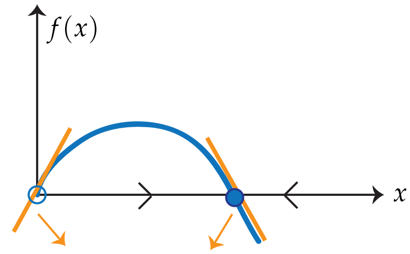
for $\tilde{x} = x - x^*$ Thus, the linearized model is:

$$\dot{\tilde{x}} = A\tilde{x}. \tag{6}$$

If $\Re\lambda_i(A) < 0$ for each eigenvalue $\lambda_i$ of A, then $x^*$ is asymp. stable.

If $\Re\lambda_i(A) > 0$ for some eigenvalue $\lambda_i$ of A, then $x^*$ is unstable.

Example: Logistic growth model above:



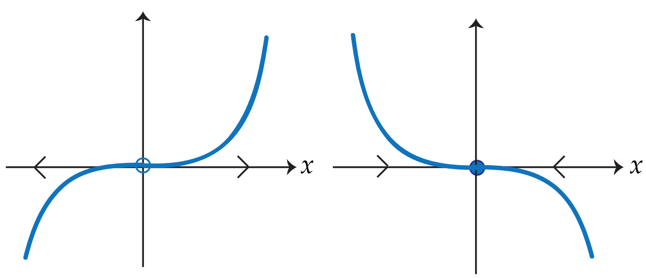

Caveats:

1. Only local properties can be determined from the linearization.

   Example: The logistic growth model linearized at $x = 0$ ($\dot{x} = rx$) would incorrectly predict unbounded growth of $x(t)$. In reality, $x(t) \to K$.

2. If $\Re\lambda_i(A) \leq 0$ with equality for some $i$, then linearization is inconclusive as a stability test. Higher order terms determine stability.

   Example: $f(x) = x^3$ vs. $f(x) = -x^3$




$f'(0) = 0$ in each case, but one is stable and the other is unstable.

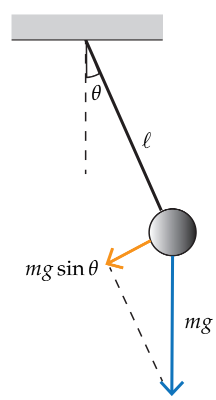
## *Motivating Example 2*

Let's consider the pendulum system with a damping coefficient $k$:

$$\ell m\ddot{\theta} = -k\ell\dot{\theta} - mg\sin\theta \tag{7}$$

or

$$\ddot{\theta} = \frac{-k}{m}\dot{\theta} - \frac{g}{l}\sin\theta \tag{8}$$



Note: These dynamics can be derived from the Lagrangian:

$$\begin{aligned}\mathcal{L}(\theta, \dot{\theta}) &= KE - PE \\ &= \frac{1}{2} m\ell^2\dot{\theta}^2 - mg\ell\cos\theta\end{aligned}$$

with the equations of motion given via the Euler-Lagrange equations (d'Alembert Principle):

The damping torque acting on the pendulum is -$\ell(k\ell\theta)$ for the planar pendulum.

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} &= \tau_{ext} \\ \frac{d}{dt}\left(m\ell^2\dot{\theta}\right) + mg\ell\sin\theta &= -k\ell^2\dot{\theta} \\ m\ell^2\ddot{\theta} + mg\ell\sin\theta &= -k\ell^2\dot{\theta} \\ \ddot{\theta} + \frac{g}{\ell}\sin\theta &= -\frac{k}{m}\dot{\theta} \\ \ddot{\theta} &= -\frac{k}{m}\dot{\theta} - \frac{g}{\ell}\sin\theta\end{aligned}$$

Define $x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$. State space: $S^1 \times \mathbb{R}$.

The system dynamics $\dot{x}$ can be rewritten in terms of this state as:

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ -\frac{k}{m}\dot{\theta} - \frac{g}{\ell}\sin\theta \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_2 - \frac{g}{\ell}\sin x_1 \end{bmatrix} \tag{9}$$

Equilibria: $(0,0)$ and $(\pi, 0)$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos x_1 & -\frac{k}{m} \end{bmatrix} = \begin{cases} \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{k}{m} \end{bmatrix} & \text{(stable) at } x_1 = 0 \\ \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & -\frac{k}{m} \end{bmatrix} & \text{(unstable) at } x_1 = \pi \end{cases}$$

Phase portrait: plot of $x_1(t)$ vs. $x_2(t)$ for 2nd order systems

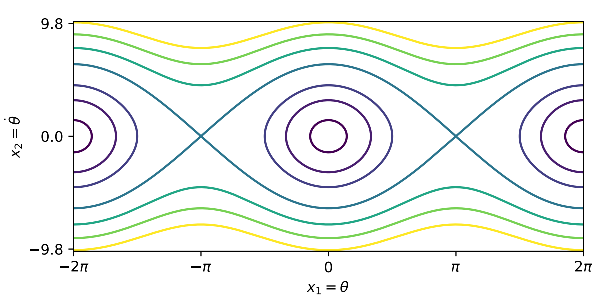



Figure 1: Phase portrait of the pendulum for the undamped case $k = 0$ with $m = 1$, $g = 9.8$, $\ell = 1$.