

ECE 6552 – Lecture 7 ¹

MATHEMATICAL BACKGROUND

January 28 2025

¹ Based on notes created by Murat Arcak and licensed under a [Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License](#).

Overview:

- Existence and Uniqueness of ODEs
- Lipschitz continuity
- Normed linear spaces
- Fixed point theorems
- Contraction mappings

Additional Reading:

- Sastry, Chapter 3
- Khalil, Chapter 3 and Appendix B

Mathematical Background

$$\dot{x} = f(x) \quad x(0) = x_0 \quad (1)$$

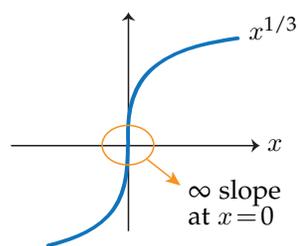
Sastry, Chapter 3

Do solutions exist? Are they unique?

- If $f(\cdot)$ is continuous (C^0) then a solution exists, but C^0 is not sufficient for uniqueness.

Example: $\dot{x} = x^{\frac{1}{3}}$ with $x(0) = 0$

$$x(t) \equiv 0, \quad x(t) = \left(\frac{2}{3}t\right)^{\frac{3}{2}} \text{ are both solutions}$$



- Sufficient condition for uniqueness: "Lipschitz continuity" (more restrictive than C^0)

$$|f(x) - f(y)| \leq L|x - y| \quad (2)$$

Definition: $f(\cdot)$ is *locally Lipschitz* if every point x^0 has a neighborhood where (2) holds for all x, y in this neighborhood for some L .

Example: $(\cdot)^{\frac{1}{3}}$ is NOT locally Lipschitz (due to ∞ slope)

$(\cdot)^3$ is locally Lipschitz:

$$x^3 - y^3 = \underbrace{(x^2 + xy + y^2)}_{\text{in any nbhd of } x^0, \text{ we can find } L \text{ to upper bound this}} (x - y)$$

in any nbhd
of x^0 , we can
find L to upper
bound this

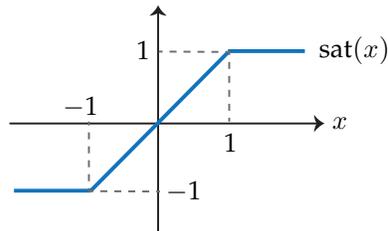
$$\implies |x^3 - y^3| \leq L|x - y|$$

- If $f(\cdot)$ is continuously differentiable (C^1), then it is locally Lipschitz.

Examples: x^3, x^2, e^x , etc.

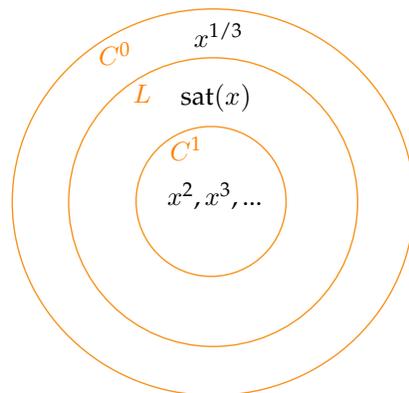
The converse is not true: local Lipschitz $\not\Rightarrow C^1$

Example:



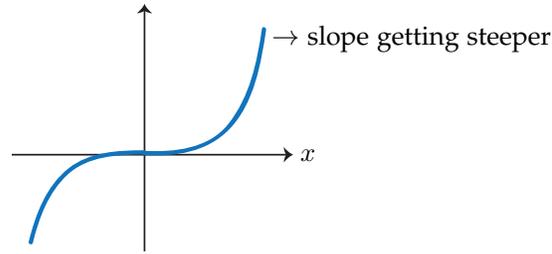
Not differentiable at $x = \mp 1$, but locally Lipschitz:

$$|\text{sat}(x) - \text{sat}(y)| \leq |x - y| \quad (L = 1).$$



Definition continued: $f(\cdot)$ is globally Lipschitz if (2) holds $\forall x, y \in \mathbb{R}^n$ (i.e., the same L works everywhere).

Examples: $\text{sat}(\cdot)$ is globally Lipschitz. $(\cdot)^3$ is not globally Lipschitz:



- Suppose $f(\cdot)$ is C^1 . Then it is globally Lipschitz iff $\frac{\partial f}{\partial x}$ is bounded.

$$L = \sup_x |f'(x)|$$

Preview of existence theorems:

1. $f(\cdot)$ is $C^0 \implies$ existence of solution $x(t)$ on finite interval $[0, t_f)$.
2. $f(\cdot)$ locally Lipschitz \implies existence and uniqueness on $[0, t_f)$.
3. $f(\cdot)$ globally Lipschitz \implies existence and uniqueness on $[0, \infty)$.

Examples:

- $\dot{x} = x^2$ (locally Lipschitz) admits unique solution on $[0, t_f)$, but $t_f < \infty$ from Lecture 1 (finite escape).
- $\dot{x} = Ax$ globally Lipschitz, therefore no finite escape

$$|Ax - Ay| \leq L|x - y| \quad \text{with} \quad L = \|A\|$$

The rest of the lecture introduces concepts that are used in proving the existence theorems mentioned above.

Normed Linear Spaces

Definition: \mathbb{X} is a normed linear space if there exists a real-valued norm $|\cdot|$ satisfying:

1. $|x| \geq 0 \quad \forall x \in \mathbb{X}, \quad |x| = 0$ iff $x = 0$.
2. $|x + y| \leq |x| + |y| \quad \forall x, y \in \mathbb{X}$ (triangle inequality)
3. $|\alpha x| = |\alpha| \cdot |x| \quad \forall \alpha \in \mathbb{R}$ and $x \in \mathbb{X}$.

Definition: A sequence $\{x_k\}$ in \mathbb{X} is said to be a Cauchy sequence if

$$|x_k - x_m| \rightarrow 0 \text{ as } k, m \rightarrow \infty. \quad (3)$$

Every convergent sequence is Cauchy. The converse is not true.

Definition: \mathbb{X} is a Banach space if every Cauchy sequence converges to an element in \mathbb{X} .

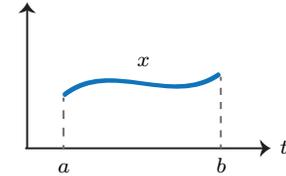
All Euclidean spaces are Banach spaces.

Example:

$C^n[a, b]$: the set of all continuous functions $[a, b] \rightarrow \mathbb{R}^n$ with norm:

$$\|x\|_C = \max_{t \in [a, b]} |x(t)|$$

1. $\|x\|_C \geq 0$ and $\|x\|_C = 0$ iff $x(t) \equiv 0$.
2. $\|x + y\|_C = \max_{t \in [a, b]} |x(t) + y(t)| \leq \max_{t \in [a, b]} \{|x(t)| + |y(t)|\} \leq \|x\|_C + \|y\|_C$
3. $\|\alpha \cdot x\|_C = \max_{t \in [a, b]} |\alpha| \cdot |x(t)| = |\alpha| \cdot \|x\|_C$



It can be shown that $C^n[a, b]$ is a Banach space.

Fixed Point Theorems

$$T(x) = x \tag{4}$$

Brouwer's Theorem (Euclidean spaces):

If U is a closed, bounded, convex subset of a Euclidean space and $T : U \rightarrow U$ is continuous, then T has a fixed point in U .

Schauder's Theorem (Brouwer's Thm \rightarrow Banach spaces):

If U is a closed bounded convex subset of a Banach space \mathbb{X} and $T : U \rightarrow U$ is *completely continuous*², then T has a fixed point in U .

² continuous and for any bounded set $B \subseteq U$ the closure of $T(B)$ is compact

Contraction Mapping Theorem:

If U is a closed subset of a Banach space and $T : U \rightarrow U$ is such that

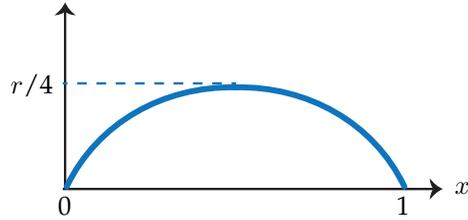
$$\|T(x) - T(y)\| \leq \rho \|x - y\| \quad \rho < 1 \quad \forall x, y \in U$$

then T has a unique fixed point in U and the solutions of $x_{n+1} = T(x_n)$ converge to this fixed point from any $x_0 \in U$.

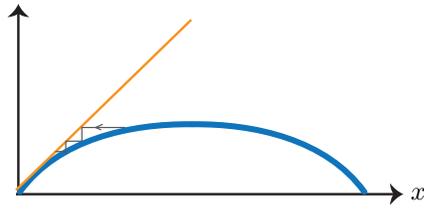
Example: The logistic map (Lecture 5)

$$T(x) = rx(1 - x) \tag{5}$$

with $0 \leq r \leq 4$ maps $U = [0, 1]$ to U . $|T'(x)| \leq r \quad \forall x \in [0, 1]$, so the contraction property holds with $\rho = r$.



If $r < 1$, the contraction mapping theorem predicts a unique fixed point that attracts all solutions starting in $[0, 1]$.



Proof steps for the Contraction Mapping Thm:

1. Show that $\{x_n\}$ formed by $x_{n+1} = T(x_n)$ is a Cauchy sequence.
Since we are in a Banach space, this implies a limit x^* exists.
2. Show that $x^* = T(x^*)$.
3. Show that x^* is unique.

Details of each step:

$$\begin{aligned}
 1. \quad |x_{n+1} - x_n| &= |T(x_n) - T(x_{n-1})| \leq \rho |x_n - x_{n-1}| \\
 &\leq \rho^2 |x_{n-1} - x_{n-2}| \\
 &\quad \vdots \\
 &\leq \rho^n |x_1 - x_0|.
 \end{aligned}$$

$$\begin{aligned}
 |x_{n+r} - x_n| &\leq |x_{n+r} - x_{n+r-1}| + \cdots + |x_{n+1} - x_n| \\
 &\leq (\rho^{n+r} + \cdots + \rho^n) |x_1 - x_0| \\
 &= \rho^n (1 + \cdots + \rho^r) |x_1 - x_0| \\
 &\leq \rho^n \frac{1}{1 - \rho} |x_1 - x_0|
 \end{aligned}$$

Since $\frac{\rho^n}{1-\rho} \rightarrow 0$ as $n \rightarrow \infty$, we have $|x_{n+r} - x_n| \rightarrow 0$ as $n \rightarrow \infty$.

$$\begin{aligned}
 2. \quad |x^* - T(x^*)| &= |x^* - x_n + T(x_{n-1}) - T(x^*)| \\
 &\leq |x^* - x_n| + |T(x_{n-1}) - T(x^*)| \\
 &\leq |x^* - x_n| + \rho |x^* - x_{n-1}|.
 \end{aligned}$$

Since $\{x_n\}$ converges to x^* , we can make this upper bound arbitrarily small by choosing n sufficiently large. This means that $|x^* - T(x^*)| = 0$, hence $x^* = T(x^*)$.

3. Suppose $y^* = T(y^*)$ $y^* \neq x^*$.

$$|x^* - y^*| = |T(x^*) - T(y^*)| \leq \rho|x^* - y^*| \implies x^* = y^*.$$

Thus we have a contradiction.