ECE 6552 – Lecture 4 [1]

PERIODIC ORBITS IN THE PLANE CONTINUED

January 16 2025

Overview:

- Bendixson's Theorem (Recall)
- Poincaré-Bendixson Theorem
- Index Theory

Additional Reading:

- Khalil, Chapter 2.6

Remarks

The Jordan form presented for linear systems with complex eigenvalues in Lecture 2 was presented as the real Jordan form. The complex Jordan form is given by:

$$J_c = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

where $\lambda_{1,2} = \alpha \pm j\beta$. For the complex form $J_c = P^{-1}AP$, P will be the eigenvectors matrix with complex entries. If we instead use the real Jordan form J_r as given in Lecture 2, then P will need to be solved accordingly to satisfy $J_r = P^{-1}AP$.

For the complex form, the similarity transformation

$$J_c = P^{-1}AP$$

holds, where P is formed from the (generally complex) eigenvectors of A.

If we instead use the real Jordan form J_r as presented in Lecture 2,

$$J_r = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix},$$

then the transformation matrix P must be constructed differently in order to satisfy

$$J_r = P^{-1}AP.$$

One way to construct such a matrix P is as follows. Let $v \in \mathbb{C}^2$ be a complex eigenvector of A corresponding to the eigenvalue $\lambda = \alpha - j\beta$, and write

$$v = u + jw, \qquad u, w \in \mathbb{R}^2.$$

Then the real matrix

$$P = \begin{bmatrix} u & w \end{bmatrix}$$

is invertible, and satisfies

$$P^{-1}AP = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} = J_r.$$

In this case, the coordinate transformation $z = P^{-1}x$ yields the real Jordan form dynamics

$$\dot{z} = J_r z.$$

Example :

Consider the system:

$$\dot{x} = \begin{bmatrix} x_1 - x_2 \\ 2x_1 - x_2 \end{bmatrix} \quad \rightarrow \quad A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$$

The eigenvalues of the system matrix are $\lambda_{1,2} = \pm j$. The real Jordan form is then:

$$J_r = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

The eigenvector corresponding to $\lambda_1 = -j$ is

$$v = \begin{bmatrix} 1 \\ 1+j \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + j \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$(A - \lambda I)v = 0$$

$$\begin{bmatrix} 1+j & -1 \\ 2 & -1+j \end{bmatrix} v = 0$$

$$\rightarrow v = \begin{bmatrix} 1 \\ 1+j \end{bmatrix}$$

Thus, we can form the transformation matrix:

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

which gives the similarity transformation:

$$J_r = P^{-1}AP = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Periodic Orbits in the Plane Continued

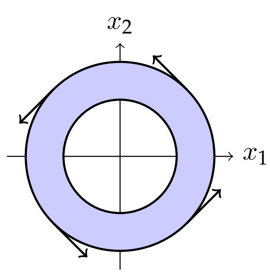
Two criteria:

1. Bendixson[2] (absence of periodic orbits)
2. Poincaré-Bendixson (existence of periodic orbits)

Poincaré-Bendixson Theorem: Suppose M is compact[3] and positively invariant for the planar, time invariant system $\dot{x} = f(x), x \in \mathbb{R}^2$. If M contains no equilibrium points, then it contains a periodic orbit.

Example: Harmonic Oscillator

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1. \end{aligned}$$

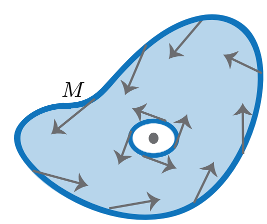


For any $R > r > 0$, the ring $\{x : r^2 \leq x_1^2 + x_2^2 \leq R^2\}$ is compact, invariant and contains no equilibria $\Rightarrow$ at least one periodic orbit. (We know there are infinitely many in this case.)

The "no equilibrium" condition in the PB theorem can be relaxed as:

"If M contains one equilibrium which is an unstable focus or unstable node"

Proof sketch: Since the equilibrium is an unstable focus or node, we can encircle it with a small closed curve on which $f(x)$ points outward. Then the set obtained from M by carving out the interior of the closed curve is positively invariant and contains no equilibrium.



Example 2, Lecture 3: Recall that we were given the system:

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= -2x_1 + x_2 - x_2(x_1^2 + x_2^2) \end{aligned}$$

We were asked to show that $B_r \triangleq \{x | x_1^2 + x_2^2 \leq r^2\}$ is positively

[2] Recall that Bendixson's theorem said that if $\text{div}(f(x)) \not\equiv 0$ and does not change sign, then D contains no periodic orbits.

[3] *i.e.*, closed and bounded

invariant for sufficiently large r:

$$\begin{aligned} f(x)\cdot n(x) &= \begin{bmatrix} x_1 + x_2 - x_1(x_1^2+x_2^2) \\ -2x_1 + x_2 - x_2(x_1^2+x_2^2) \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= x_1^2 + x_1x_2 - x_1^2(x_1^2+x_2^2) - 2x_1x_2 + x_2^2 - x_2^2(x_1^2+x_2^2) \\ &= -x_1x_2 + (x_1^2+x_2^2) - (x_1^2+x_2^2)^2 \end{aligned}$$

which used the inequality

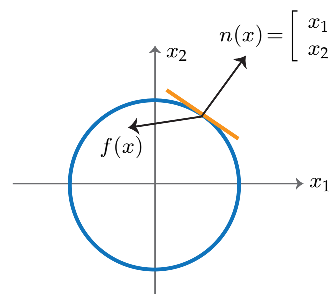
$$|2x_1x_2| \leq x_1^2 + x_2^2,$$

This is a special case of the Cauchy-Schwarz inequality: $|\langle a,b\rangle| \leq \|a\|\|b\|$ with $a=(x_1,x_2)$ and $b=(x_2,x_1)$:

$$\begin{aligned} |x_1x_2 + x_2x_1| &\leq \sqrt{(x_1^2+x_2^2)(x_1^2+x_2^2)} \\ |2x_1x_2| &\leq x_1^2+x_2^2 \end{aligned}$$

to arrive at the final condition:

$$\begin{aligned} f(x)\cdot n(x) &\leq \frac{1}{2}(x_1^2+x_2^2) + (x_1^2+x_2^2) - (x_1^2+x_2^2)^2 \\ &= \frac{3}{2}r^2 - r^4 \end{aligned}$$



Therefore, $f(x)\cdot n(x) \leq \frac{3}{2}r^2 - r^4 \leq 0$ if $r^2 \geq \frac{3}{2}$.

So, we can conclude that B_r is positively invariant for $r \geq \sqrt{\frac{3}{2}}$ and contains the equilibrium $x = 0$.

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \qquad \lambda_{1,2} = 1 \mp j\sqrt{2} \qquad \text{unstable focus.}$$

Therefore, B_r must contain a periodic orbit.

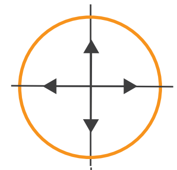
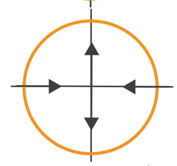
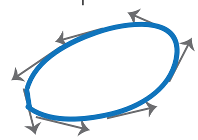
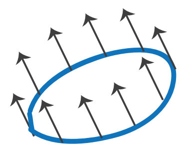
A more general form of the PB Theorem states that, for time invariant, planar systems, bounded trajectories converge to equilibria, periodic orbits, or unions of equilibria connected by trajectories.

Corollary: No chaos for time invariant planar systems.

Index Theory

Again, applicable only to planar systems.

Definition (index): The index of a closed curve is k if, when traversing the curve in one direction, $f(x)$ rotates by $2\pi k$ in the same direction. The index of an equilibrium is defined to be the index of a small curve around it that doesn't enclose another equilibrium.

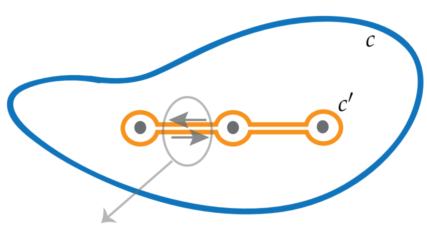
type of equilibrium or curve	index	
node, focus, center	+1	
saddle	-1	
any closed orbit	+1	
a closed curve not encircling any equilibria	0	

The last claim (index $= 0$) follows from the following observations:

- Continuously deforming a closed curve without crossing equilibria leaves its index unchanged.
- A curve not encircling equilibria can be shrunk to an arbitrarily small one, so $f(x)$ can be considered constant.

Theorem: The index of a closed curve is equal to the sum of indices of the equilibria inside.

Graphical proof: Shrinking curve c to c' below without crossing equilibria does not change the index. The index of c' is the sum of the indices of the curves encircling the equilibria because the thin "pipes" connecting these curves do not affect the index of c'.



contributions from the two sides cancel out

The following corollary is useful for ruling out periodic orbits (like Bendixson's Theorem studied in the previous lecture):

Corollary: Inside any periodic orbit there must be at least one equilibrium and the indices of the equilibria enclosed must add up to $+1$.

Example (from last lecture):

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\delta x_2 + x_1 - x_1^3 + x_1^2 x_2 \quad \delta > 0\end{aligned}$$

Bendixson's Criterion: No periodic orbit can lie entirely in one of the regions $x_1 \leq -\sqrt{\delta}$, $-\sqrt{\delta} \leq x_1 \leq \sqrt{\delta}$, or $x_1 \geq \sqrt{\delta}$.

Now apply the corollary above.

Equilibria: $(0,0)$, $(\mp 1, 0)$. To find their indices evaluate the Jacobian:

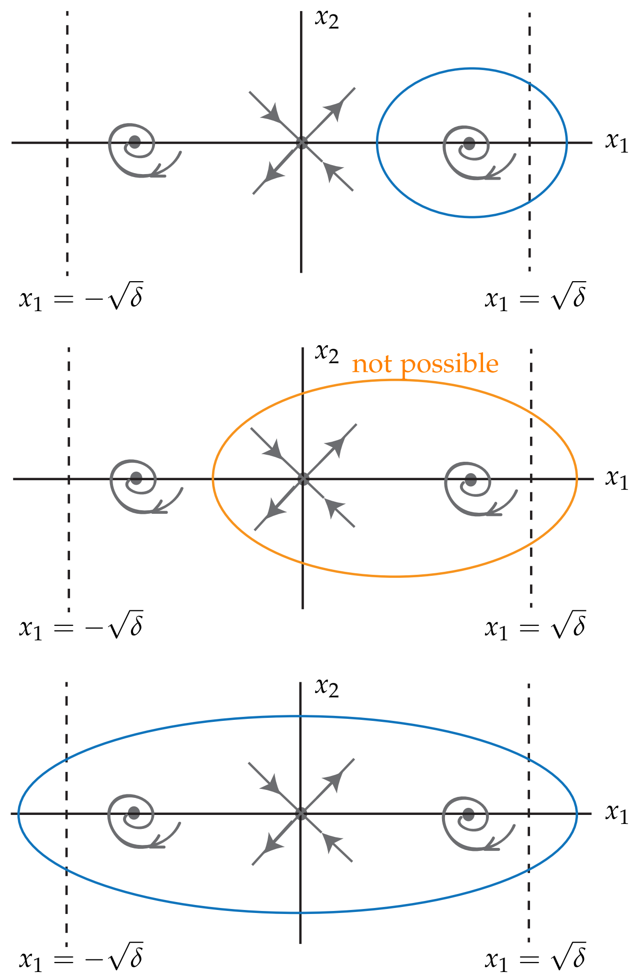
$$\left.\frac{\partial f}{\partial x}\right|_{x=(0,0)} = \begin{bmatrix} 0 & 1 \\ 1 & -\delta \end{bmatrix} \qquad \lambda^2 + \delta\lambda \underbrace{-1}_{<0} = 0.$$

The eigenvalues are real and have opposite signs, therefore $(0,0)$ is a saddle: index $= -1$.

$$\left.\frac{\partial f}{\partial x}\right|_{x=(\mp 1,0)} = \begin{bmatrix} 0 & 1 \\ -2 & 1-\delta \end{bmatrix} \qquad \lambda^2 + (\delta - 1)\lambda \underbrace{+2}_{>0} = 0.$$

The eigenvalues are either real with the same sign (node) or complex conjugates (focus or center), therefore $(\mp 1, 0)$ each has index$= +1$.

Thus, the corollary above rules out the periodic orbit in the middle plot below. It does not rule out the others, but does not prove their existence either. Bendixson's Criterion rules out neither of the three.


x_2
x_1
$x_1 = -\sqrt{\delta}$
$x_1 = \sqrt{\delta}$
x_2
not possible
x_1
$x_1 = -\sqrt{\delta}$
$x_1 = \sqrt{\delta}$
x_2
x_1
$x_1 = -\sqrt{\delta}$
$x_1 = \sqrt{\delta}$