ECE 6552 – Lecture 3[1]

PHASE POTRAITS OF NONLINEAR SYSTEMS NEAR HYPERBOLIC EQUILIBRIA

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Overview:

- Hartman-Grobman Theorem
- Bendixson's Theorem
- Invariant Sets

Additional Reading:

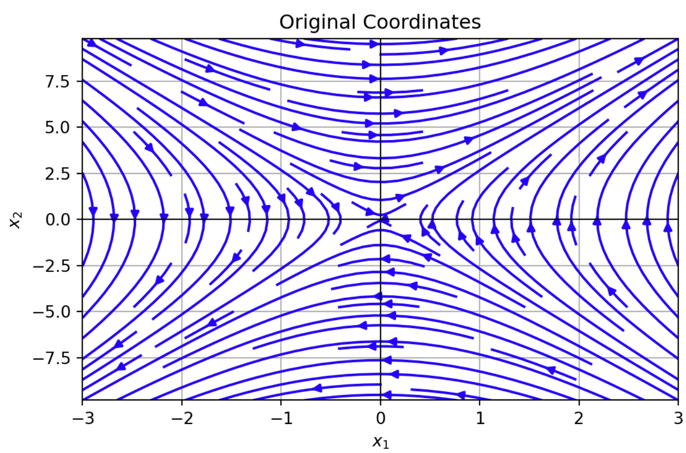
- Khalil, Chapter 2

Review: Phase Portraits of Linear Systems: $\dot{x} = Ax$

Consider our pendulum linearized at the upright angle

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{k}{m} \end{bmatrix} x$$

Let's specifically take $g = 9.8$, $k = 0$, $m = 1$, and $l = 1$. The eigenvalues for the system are then $\lambda_1 = 3.13$, $\lambda_2 = -3.13$. From yesterday, we know that this yields a saddle node, but we can further illustrate this using the phase portrait:



We can transform this into Jordan Form by first setting our Jordan form matrix to:

$$J = \begin{bmatrix} 3.13 & 0 \\ 0 & -3.13 \end{bmatrix}$$

which yields the eigenvectors:

These eigenvectors are found by solving $(A - \lambda_i I)v_i = 0$

$$v_1 = \begin{bmatrix} 1 \\ 3.13 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ -3.13 \end{bmatrix}$$

Thus the transformation into jordan form is provided by $J = P^{-1}AP$ with the matrix:

$$P = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3.13 & -3.13 \end{bmatrix}$$

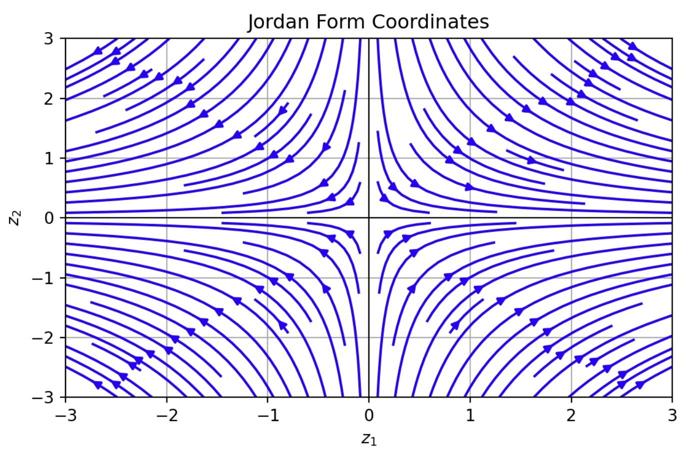
We can recalculate J as a sanity check. Finaly, we can transform our coordinates using the transformation:

$$z = P^{-1}x$$

with the dynamics

$$\dot{z}_1 = \lambda_1 z_1, \quad \dot{z} = \lambda_2 z_2$$

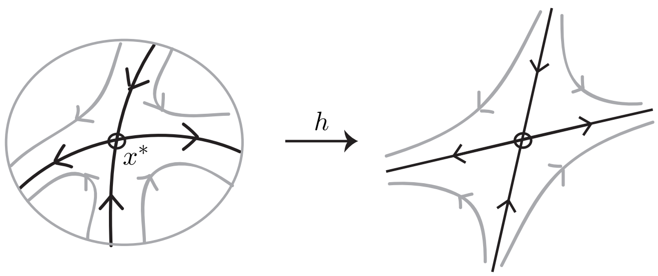
This new system yields the phase portrait shown below



Phase Portraits of Nonlinear Systems Near Hyperbolic Equilibria

Definition: *Hyperbolic Equilibrium.* Linearization has no eigenvalues on the imaginary axis

Phase portraits of nonlinear systems near hyperbolic equilibria are qualitatively similar to the phase portraits of their linearization. According to the Hartman-Grobman Theorem (below) a "continuous deformation" maps one phase portrait to the other.



Theorem: Hartman-Grobman Theorem.
If x^* is a hyperbolic equilibrium of $\dot{x} = f(x), x \in \mathbb{R}^n$, then there exists a *homeomorphism*[2] $z = h(x)$ defined in a neighborhood of x^* that maps trajectories of $\dot{x} = f(x)$ to those of $\dot{z} = Az$ where $A \triangleq \frac{\partial f}{\partial x}\big|_{x=x^*}$.

[2] a continuous map with a continuous inverse

The hyperbolicity condition can't be removed:

<u>Example:</u>

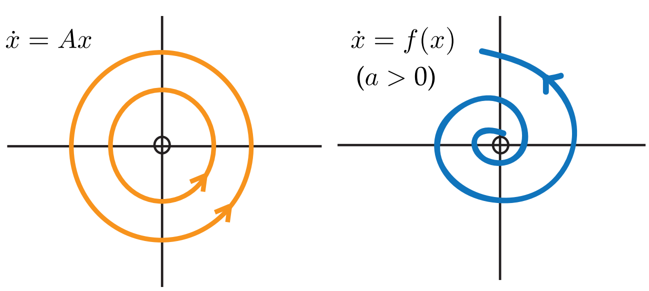
$$\begin{aligned} \dot{x}_1 &= -x_2 + ax_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= x_1 + ax_2(x_1^2 + x_2^2) \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \dot{r} &= ar^3 \\ \dot{\theta} &= 1 \end{aligned}$$

This can be equivalently written in vector form as

$$\dot{x} = \begin{bmatrix} -x_2 + ax_1(x_1^2 + x_2^2) \\ x_1 + ax_2(x_1^2 + x_2^2) \end{bmatrix}$$

$$x^* = (0,0) \qquad A = \frac{\partial f}{\partial x}\bigg|_{x=x^*} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

There is no continuous deformation that maps the phase portrait of the linearization to that of the original nonlinear model:



Periodic Orbits in the Plane

Theorem: Bendixson's Theorem. For a time-invariant planar system

$$\dot{x}_1 = f_1(x_1, x_2) \quad \dot{x}_2 = f_2(x_1, x_2),$$

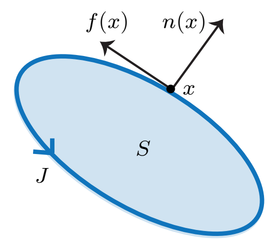
if the divergence $\nabla \cdot f(x) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ is not identically zero and does not change sign in a simply connected region D, then there are no periodic orbits lying entirely in D.

$$\begin{aligned}\text{div}(f(x)) &= \nabla \cdot f(x) \\ &= \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right) \cdot (f_1, \cdots, f_n) \\ &= \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}.\end{aligned}$$

Note: the form $\nabla \cdot f(x)$ is sometimes considered an abuse of notation since we should not apply an operator (∇) through multiplication.

Proof: By contradiction. Suppose a periodic orbit J lies in D. Let S denote the region enclosed by J and $n(x)$ the normal vector to J at x. Then $f(x) \cdot n(x) = 0$ for all $x \in J$. By the Divergence Theorem:

$$\underbrace{\int_J f(x) \cdot n(x) d\ell}_{=0} = \underbrace{\iint_S \nabla \cdot f(x) dx}_{\neq 0}.$$



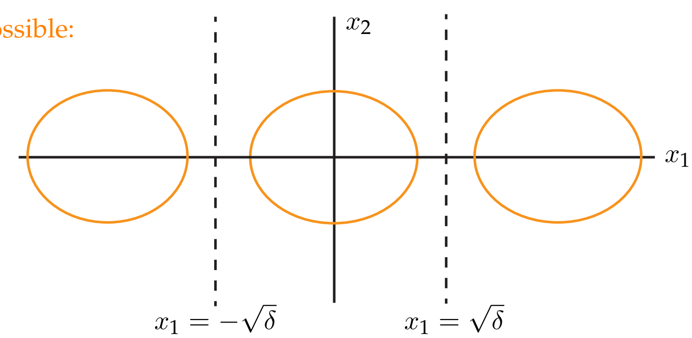
Example:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\delta x_2 + x_1 - x_1^3 + x_1^2 x_2 \quad \delta > 0\end{aligned}$$

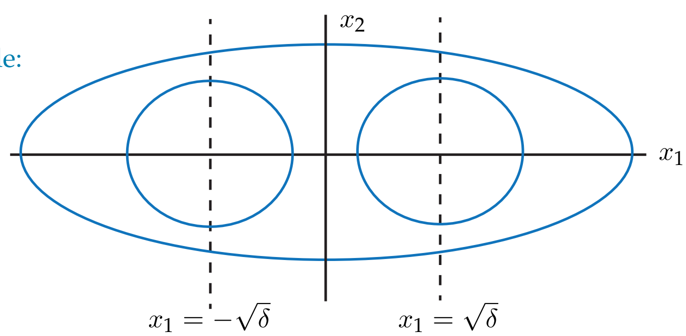
$$\nabla \cdot f(x) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = x_1^2 - \delta$$

Therefore, no periodic orbit can lie entirely in the region $x_1 \leq -\sqrt{\delta}$ where $\nabla \cdot f(x) \geq 0$, or $-\sqrt{\delta} \leq x_1 \leq \sqrt{\delta}$ where $\nabla \cdot f(x) \leq 0$, or $x_1 \geq \sqrt{\delta}$ where $\nabla \cdot f(x) \geq 0$.

not possible:



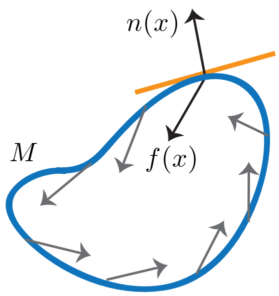
possible:



Invariant Sets

Notation: $\varphi(t, x_0)$ denotes a trajectory of $\dot{x} = f(x)$ with initial condition $x(0) = x_0$.

Definition: A set $M \subset \mathbb{R}^n$ is positively (negatively) invariant if, for each $x_0 \in M$, $\varphi(t, x_0) \in M$ for all $t \geq 0$ ($t \leq 0$).



If $f(x) \cdot n(x) \leq 0$ on the boundary then M is positively invariant.

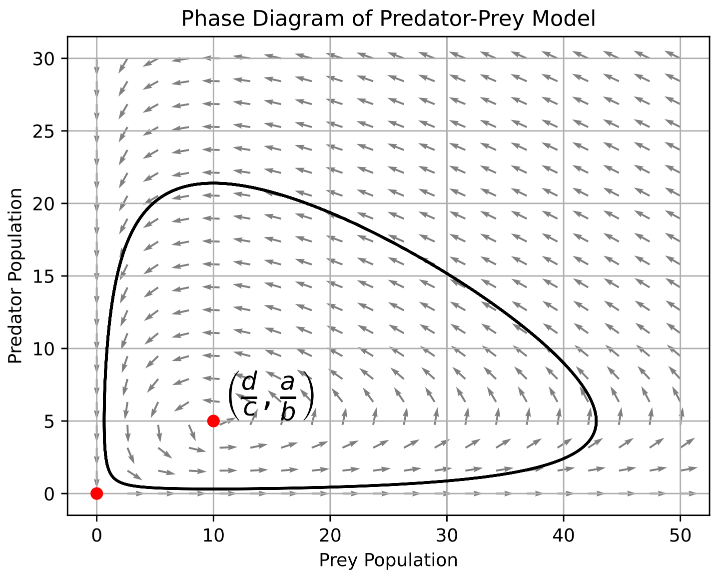
Example 1: A predator-prey model (Lotka-Volterra equations)

$$
\begin{aligned}
\dot{x} &= (a - by)x && \text{Prey (exponential growth when } y = 0) \\
\dot{y} &= (cx - d)y && \text{Predator (exponential decay when } x = 0) \\
& a, b, c, d, > 0
\end{aligned}
$$

The nonnegative quadrant is invariant:

$$
\text{(x-axis:)} \quad \begin{bmatrix} ax \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} = 0
$$

$$
\text{(y-axis:)} \quad \begin{bmatrix} 0 \\ -dy \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix} = 0
$$



Example 2: (Similar to Example 2.8 in Khalil)

$$\dot{x}_1 = x_1 + x_2 - x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = -2x_1 + x_2 - x_2(x_1^2 + x_2^2)$$

Show that $B_r \triangleq \{x | x_1^2 + x_2^2 \leq r^2\}$ is positively invariant for sufficiently large r.

$$\begin{aligned} f(x) \cdot n(x) &= \begin{bmatrix} x_1 + x_2 - x_1(x_1^2 + x_2^2) \\ -2x_1 + x_2 - x_2(x_1^2 + x_2^2) \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= x_1^2 + x_1x_2 - x_1^2(x_1^2 + x_2^2) - 2x_1x_2 + x_2^2 - x_2^2(x_1^2 + x_2^2) \\ &= -x_1x_2 + (x_1^2 + x_2^2) - (x_1^2 + x_2^2)^2 \end{aligned}$$

Next, we can use the inequality

$$|2x_1x_2| \leq x_1^2 + x_2^2,$$

This is a special case of the Cauchy-Schwarz inequality: $|\langle a, b\rangle| \leq \|a\|\|b\|$ with $a = (x_1, x_2)$ and $b = (x_2, x_1)$:

$$|x_1x_2 + x_2x_1| \leq \sqrt{(x_1^2 + x_2^2)(x_1^2 + x_2^2)}$$
$$|2x_1x_2| \leq x_1^2 + x_2^2$$

to arrive at the final condition:

$$\begin{aligned} f(x) \cdot n(x) &\leq \frac{1}{2}(x_1^2 + x_2^2) + (x_1^2 + x_2^2) - (x_1^2 + x_2^2)^2 \\ &= \frac{3}{2}r^2 - r^4 \end{aligned}$$

Therefore, $f(x) \cdot n(x) \leq \frac{3}{2}r^2 - r^4 \leq 0$ if $r^2 \geq \frac{3}{2}$.