ECE 6552 – Lecture 25

EXAM REVIEW

April 16 2026

Overview:

- Feedback Linearization
- Normal Form and Zero Dynamics
- Control Lyapunov functions
- Control Barrier functions

Additional Reading:

- Khalil Chapter 13 (Feedback Linearization of SISO Systems)
- Sastry Chapter 9.3 (Feedback Linearization of MIMO Systems)
- E. Sontag, 1983 (Control Lyapunov Functions)
- A. Ames et al. 2019 (Control Barrier Functions)

Feedback Linearization

Relative Degree

Definition: *Relative Degree for SISO.* A SISO system has relative degree r if, in a neighborhood of the equilibrium:

$$L_g L_f^{i-1} h(x) = 0, \quad i = 1, 2, \dots, r-1$$
$$L_g L_f^{r-1} h(x) \neq 0$$

Informally, this is the same as saying that "A SISO system has relative degree r if the input does not appear until the r-th derivative of the output $h(x)$".

Definition: *Relative Degree for MIMO.* A MIMO system has relative degree r_i for each output $h_i(x)$ if the i-th output needs to be differentiated r_i times before *some* input appears.

Definition: *Vector Relative Degree for MIMO.* A MIMO system has vector relative degree $r = \{r_1, \dots, r_m\}$ if the matrix $A(x)$ is nonsingular:

$$A(x) = \begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & \cdots & L_{g_m} L_f^{r_1-1} h_1(x) \\ \vdots & \ddots & \vdots \\ L_{g_1} L_f^{r_m-1} h_m(x) & \cdots & L_{g_m} L_f^{r_m-1} h_m(x) \end{bmatrix}$$

Example 1: Consider the system:

$$\dot{x}_1 = x_1$$
$$\dot{x}_2 = x_2 + u$$
$$y = x_1$$

The system does not have a well-defined relative degree because $\dot{y} = \dot{x}_1 = x_1 = y$. Thus the input u will *never* appear.

Example 2: Consider the system:

$$\dot{x}_1 = -x_1 + \frac{2 + x_3^2}{1 + x_3^2} u$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = x_1 x_3 + u$$
$$y = x_2$$

The system has relative degree 2 because:

$$\dot{y} = \dot{x}_2 = x_3$$
$$\ddot{y} = \dot{x}_3 = x_1 x_3 + u$$

Notably, the relative degree is well-defined for all $x \in \mathbb{R}^3$.

Example 3: Consider the system (it is the controlled van der Pol equation):

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2 + u$$
$$y = x_2$$

The system has relative degree 1 because $\dot{y} = \dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2 + u$. This is also well-defined for all $x \in \mathbb{R}^2$.

Example 4: Consider the MIMO system:

$$\dot{x}_1 = \cos(x_3) u_1$$
$$\dot{x}_2 = \sin(x_3) u_1$$
$$\dot{x}_3 = u_2$$
$$y_1 = x_1$$
$$y_2 = x_2$$

The system has relative degree $r_1 = r_2 = 1$ because

$$\dot{y}_1 = \dot{x}_1 = \cos(x_3) u_1$$
$$\dot{y}_2 = \dot{x}_2 = \sin(x_3) u_1$$

To check if the system has a valid vector relative degree, we need to check if the matrix $A(x)$ is nonsingular. Explicitly, this matrix is:

$$A := \begin{bmatrix} \cos(x_3) & 0 \\ \sin(x_3) & 0 \end{bmatrix}$$

This matrix is NOT nonsingular, so it does not have a valid vector relative degree. This means that we could not perform feedback linearization on this system. Instead, we would need to perform dynamic extension:

$$\begin{aligned} \dot{x}_1 &= x_4 \cos(x_3) \\ \dot{x}_2 &= x_4 \sin(x_3) \\ \dot{x}_3 &= u_2 \\ \dot{x}_4 &= u_1 \\ y_1 &= x_1 \\ y_2 &= x_2 \end{aligned}$$

This would result in each output having relative degree 2, with the derivatives being:

$$\begin{aligned} \dot{y}_1 &= \dot{x}_1 = x_4 \cos(x_3) \\ \ddot{y}_1 &= u_1 \cos(x_3) - x_4 \sin(x_3) u_2 \\ \\ \dot{y}_2 &= \dot{x}_2 = x_4 \sin(x_3) \\ \ddot{y}_2 &= u_1 \sin(x_3) + x_4 \cos(x_3) u_2 \end{aligned}$$

Thus, the A matrix is now:

$$A := \begin{bmatrix} \cos(x_3) & -x_4 \sin(x_3) \\ \sin(x_3) & x_4 \cos(x_3) \end{bmatrix}$$

This matrix is only singular when $x_4 = 0$, so for any state such that $x_4 \neq 0$, the system has a valid vector relative degree $r = \{2, 2\}$.

Input-Output Linearization

If a system has a well-defined realtive degree (or a valid vector relative degree for MIMO systems) then it is input-output linearizable. Explicitly, this feedback linearizing control law is:

$$u = \frac{1}{L_g L_f^{r-1} h(x)} \left(-L_f^r h(x) + v \right)$$

or

$$u = A^{-1}(-B + v)$$

You can always think of this as the latter if you rearrange the system to be in the form:

$$y^{(r)} = B + Au$$

By selecting the auxiliary control law

$$v = -k_1 y - k_2 \dot{y} - \cdots - k_r y^{(r-1)} \tag{1}$$

we can transform our input-output system to be:

$$\begin{bmatrix} \dot{y} \\ \ddot{y} \\ \vdots \\ y^{(r)} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -k_1 & -k_2 & -k_3 & \cdots & -k_r \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \\ \ddots \\ y^{(r-1)} \end{bmatrix}$$

Full-State Feedback Linearization

If $r = n$, then there exists a diffeomorphism that transforms the system into the linear system

$$\dot{\eta} = A\eta$$

with the transformation being:

$$x \mapsto \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{bmatrix} = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix}$$

We have a theorem to verify when a system is provably full-state feedback linearizable. This theorem also provides us with tools to know how to select the output $h(x)$ such that the system is full-state feedback linearizable.

Theorem: Full-State Feedback Linearizable. *The system $\dot{x} = f(x) + g(x)u$ is full-state feedback linearizable around x_0 if and only if the following two conditions hold:*

C1) $\begin{bmatrix} g(x_0) & \mathrm{ad}_f\, g(x_0) & \dots & \mathrm{ad}_f^{n-1}\, g(x_0) \end{bmatrix}$ *has rank n.*

C2) The distribution $\Delta(x) = \mathrm{span}\{g(x), \mathrm{ad}_f\, g(x), \dots, \mathrm{ad}_f^{n-2}\, g(x)\}$ is involutive in a neighborhood of x_0.

Importantly, by the Frobenius theorem, a nonsingular distribution is involutive if and only if it is completely integrable, which gives us the condition that there must exist a function $h(x)$ such that:

$$\frac{\partial h}{\partial x} f_j = 0$$

where f_j represents each element in the span of the associated distribution Δ.

Example 5: Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sin(x_1) + u\end{aligned}$$

First, to calculate the adjoint elements:

$$\begin{aligned}g(x) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \mathrm{ad}_f\, g(x) = [f, g] &= \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x) = -\begin{bmatrix} 0 & 1 \\ \cos(x_1) & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\end{aligned}$$

Thus, the matrix of condition 1 is:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

which is full rank.

Second, we need to find an output $h(x)$ such that

$$\frac{\partial h}{\partial x} g(x) = 0 \implies \frac{\partial h}{\partial x} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$

This condition is satisfied for $h(x) = x_1$.

We can double check this by computing the relative degree associated with $h(x) = x_1$:

$$\begin{aligned}\dot{y} &= \dot{x}_1 = x_2 \\ \ddot{y} &= \dot{x}_2 = \sin(x_1) + u\end{aligned}$$

Zero Dynamics and Normal Form

If the system is *not* full-state feedback linearizable, the system will have *zero dynamics*. The zero dynamics are those that remain when the feedback linearizing control law is applied (with $v = 0$) and the outputs are consequently driven to zero.

Example 6: Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + x_3^2 + u \\ \dot{x}_3 &= -x_3 + x_1 \\ y &= x_1\end{aligned}$$

First, we analyze the relative degree of the system:

$$\begin{aligned}\dot{y} &= \dot{x}_1 = x_2 \\ \ddot{y} &= \dot{x}_2 = -x_1 + x_3^2 + u\end{aligned}$$

Thus, the system has relative degree $r = 2$. The associated outputs are $y = x_1$ and $\dot{y} = x_2$. The feedback control law is:

$$u = x_1 - x_3^2 + v$$

The zero dynamics can then be derived as:

$$\begin{aligned}\dot{x}_1 &= 0 \\ \dot{x}_2 = 0 + x_3^2 + (0 - x_3^2 + 0) &= 0 \\ \dot{x}_3 &= -x_3 + 0\end{aligned}$$

Thus, the zero dynamics are $\dot{x}_3 = -x_3$.

To derive the zero dynamic coordinate transformation, we must find the transformation z such that z is independent of the outputs, and $\dot{z}$ does not contain u. This is done by ensuring that $\nabla z \cdot g(x) = 0$.

Example 6 continued: The zero dynamic coordinates associated with our previous example can be derived by finding z to satisfy:

$$\frac{\partial z}{\partial x}\begin{bmatrix}0\\1\\0\end{bmatrix} = 0 \quad \Longrightarrow \quad z = x_3$$

Thus, our transformation to normal form is:

$$T : x \mapsto \begin{bmatrix}\eta_1\\ \eta_2\\ z\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

Note: This example is trivial since the normal form is already decomposed as exactly our system state...

The full normal form dynamics are:

$$\begin{bmatrix}\dot{\eta}_1\\ \dot{\eta}_2\\ \dot{z}\end{bmatrix} = \begin{bmatrix}\eta_2\\ v\\ -z\end{bmatrix}$$

We can check whether this map is a diffeomorphism (with a smooth inverse) by if its Jacobian has full rank.

Note: You should check the Jacobian if a question asks you to "specify the region over which the transformation to Normal Form is valid"

Example 8 continued: The Jacobian of the transformation is:

$$DT = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since this is full rank, our transformation is a diffeomorphism for all $x \in \mathbb{R}^3$.

Control Lyapunov functions

Definition: *Control Lyapunov Function.* A positive definite function $V(x)$ is a (global) control Lyapunov function for the system $\dot{x} = f(x) + g(x)u$ if $\forall x \neq 0$, $\exists u$ such that:

$$\dot{V}(x) = \frac{\partial V}{\partial x}\left(f(x) + g(x)u\right) < 0$$

One approach is to use Sontag's formula which is a closed-form solution to our inequality condition:

$$u = \begin{cases} -\left(\left(\frac{\partial V}{\partial x} f\right) + \sqrt{\left(\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4 \right)} \right) \Big/ \left(\frac{\partial V}{\partial x} g\right) & \textit{if } \frac{\partial V}{\partial x} g \neq 0 \\ 0 & \textit{if } \frac{\partial V}{\partial x} g = 0 \end{cases}$$

The alternative approach is to use convex optimization to solve the problem:

$$\begin{aligned} u^* = \underset{\mu}{\text{minimize}} \quad & \|\mu\|^2 \\ \text{subject to} \quad & L_f V(x) + L_g V(x)\mu < 0 \end{aligned}$$

Control Barrier functions

The summary of control Lyapunov functions compared to control barrier functions is:

$$\underbrace{\dot{V} \leq -\alpha(V(x))}_{\text{Stability}} \quad \text{versus} \quad \underbrace{\dot{h} \geq -\alpha(h(x))}_{\text{Safety}}$$

Definition: *Barrier Function.*A function h with $\mathcal{C} = \{x \mid h(x) \geq 0\}$ is a barrier function for $\dot{x} = f(x)$ if there exists a locally Lipschitz function $\alpha : \mathbb{R} \to \mathbb{R}$ satisfying $\alpha(0) = 0$ such that

$$\dot{h}(x) \geq -\alpha(h(x)), \quad \text{for all } x \in \mathbb{R}^n$$

Definition: *Control Barrier Function.*A function h with $\mathcal{C} = \{x \mid h(x) \geq 0\}$ is a control barrier function for $\dot{x} = f(x) + g(x)u$ if there exists a locally Lipschitz function $\alpha : \mathbb{R} \to \mathbb{R}$ satisfying $\alpha(0) = 0$ such that

$$\sup_{u \in \mathbb{R}^m} \dot{h}(x) \geq -\alpha(h(x)), \quad \text{for all } x \in \mathbb{R}^n$$

As with control Lyapunov functions, we can use either a closed-form expression or convex optimization to find an input that satisfies our inequality condition. The closed-form expression is:

$$u = \begin{cases} 0 & \textit{if } L_f h + \alpha(h(x)) \geq 0 \\ \frac{-(L_f h + \alpha(h(x))) L_g h^T}{\|L_g h\|^2} & \textit{otherwise} \end{cases}$$

The convex optimization approach can take many forms, but we discussed two main ones. The minimum effort control barrier function is:

$$\begin{aligned} u^* = \underset{\mu}{\text{minimize}} \ & \|\mu\|^2 \\ \text{subject to} \quad & L_f h(x) + L_g h(x)\mu \geq -\alpha(h(x)) \end{aligned}$$

The minimally-invasive control barrier function is:

$$\begin{aligned} u^* = \underset{\mu}{\text{minimize}} \ & \|\mu - k(x)\|^2 \\ \text{subject to} \quad & L_f h(x) + L_g h(x)\mu \geq -\alpha(h(x)) \end{aligned}$$

Lastly, if $L_g h(x) \equiv 0$, we will need to instead use a higher-order barrier function.

Example 7: Consider the system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -x_1^2 + u \end{aligned}$$

Synthesize a control barrier function to keep the state x_1 below a threshold of 2.

This desired behavior can be encoded by the function $h(x) = 2 - x_1 \geq 0$. This is associated with the safe setlength

$$\mathcal{C} = \{x \in \mathbb{R}^3 \mid h(x) = 2 - x_1 \geq 0\}$$

Taking the derivative, we get:

$$\dot{h} = -\dot{x}_1 = -x_2 \qquad \Longrightarrow \qquad L_f h = -x_2,\ L_g h = 0$$

Thus, this is an invalid control barrier function because $L_g h \equiv 0$.

Instead, we will need to use a higher-order barrier function of the form:

$$\Psi(x) := \dot{h}(x) + \alpha(h(x))$$

which is associated with its own safe set

$$\mathcal{C}_1 = \{x \in \mathbb{R}^3 \mid \Psi(x) \geq 0\}$$

To check if this higher order barrier function is valid. While doing this, we will assume $\alpha(s) = \gamma_1 s$ for simplicity.

$$\begin{aligned}
\dot{\Psi} &= \ddot{h}(x) + \alpha'(h(x))\dot{h} \\
&= -\dot{x}_2 + \gamma_1(-x_2) \\
&= -x_3 - \gamma_1 x_2
\end{aligned}$$

Since this is *still* not valid, we will need to take another higher-order derivative:

$$\Psi_2(x) = \dot{\Psi}(x) + \alpha_2(\Psi(x))$$

which is associated with the safe set

$$\mathcal{C}_2 = \{x \in \mathbb{R}^3 \mid \Psi_2(x) \geq 0\}$$

To check if this higher order barrier function is valid, we need to again check whether $L_g\Psi_2 \neq 0$. Again, we will assume $\alpha_2(s) = \gamma_2 s$ for simplicity.

$$\begin{aligned}
\dot{\Psi}_2 &= \ddot{\Psi}(x) + \alpha_2'(\Psi(x))\dot{\Psi} \\
&= (-\dot{x}_3 - \gamma_1 \dot{x}_2) + \gamma_2(-x_3 - \gamma_1 x_2) \\
&= -(-x_1^2 + u) - \gamma_1 x_3 + \gamma_2(-x_3 - \gamma_1 x_2)
\end{aligned}$$

Here, $L_g\Psi_2 = -1$ which means that the higher-order barrier function is valid everywhere. Finally, we will enforce this higher-order barrier function by finding u such that:

$$L_f\Psi_2(x) + L_g\Psi_2(x)u \geq -\alpha_2(\Psi_2(x))$$