# *ECE 6552 – Lecture 21*

## CONVEX OPTIMIZATION PROBLEMS

*April 2 2026*

Overview:

- Motivating examples: CLF-based controllers and Lyapunov search
- Convex functions and sets
- Convex optimization problems and optimality conditions
- Important classes: LP, QP, QCQP, SOCP, SDP
- Linear matrix inequalities (LMIs)
- Solving convex optimization problems in practice

Additional Reading:

- S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.

## *Motivating Examples*

Before developing the theory, we look at two problems from nonlinear control that are naturally cast as convex optimization problems.

### *Minimum-Effort Stabilization via CLF*

Given the system $\dot{x} = f(x) + g(x)u$ and a Control Lyapunov Function (CLF) $V(x)$, we can synthesize a minimum-effort controller by solving

$$k(x) = \arg\min_{u} \ \|u\|^2 \quad \text{subject to} \quad \frac{\partial V}{\partial x}(f(x) + g(x)u) \leq -\varepsilon V(x), \tag{1}$$

where $\varepsilon > 0$ is user-chosen. We use $\leq -\varepsilon V(x)$ rather than a strict inequality because strict inequality constraints are not tractable in optimization. This is a **quadratic program (QP)**: the cost is quadratic and the CLF constraint is affine in $u$.

### *Finding Polynomial Lyapunov Functions*

Given $\dot{x} = f(x)$, we can search for a Lyapunov function by solving

$$c^* = \arg\min_{c} \ 0 \quad \text{s.t.} \quad V(x) \geq \varepsilon_1 A(x) \ \forall x, \quad \frac{\partial V}{\partial x} f(x) \leq -\varepsilon_2 V(x) \ \forall x, \tag{2}$$

e.g., for $x \in \mathbb{R}^2$, $V(x) = c_1 x_1^4 + c_2 x_1^3 x_2 + \cdots + c_n$. Here, $A(x)$ is a positive definite function and the zero cost means this is a **feasibility problem**. The "$\forall x$" quantifier introduces infinitely many constraints, which are handled in practice via SOS or SDP relaxations.

## *Optimization Problems*

We consider problems of the form

$$\begin{aligned} \text{minimize}_x \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0, \quad i = 1, \ldots, m \end{aligned} \tag{3}$$

To maximize $\tilde{f}_0(x)$, set $f_0 = -\tilde{f}_0$. Equality constraints $h(x) = 0$ are encoded as $h(x) \leq 0$ and $-h(x) \leq 0$.

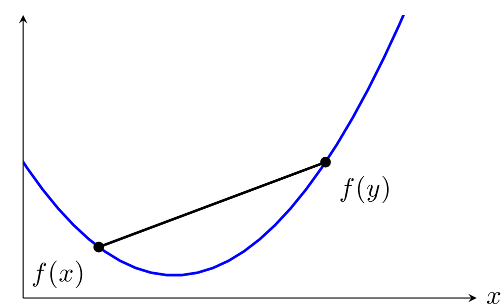
where $x \in \mathbb{R}^n$ is the optimization variable, $f_0$ is the objective function, and $f_i(x)$ are constraint functions.

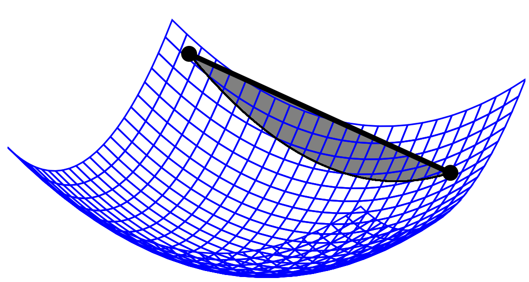
The **optimal value** is the smallest value of $f_0$ on the feasible set; a point achieving it is an **optimal point**.

## *Convex Functions and Sets*

**Definition**: *Convex function.* $f : \mathbb{R}^n \to \mathbb{R}$ is convex if for all $x, y$ and all $0 \leq \theta \leq 1$:

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y).$$



**First- and second-order tests.** When $f$ is once differentiable, $f$ is convex iff

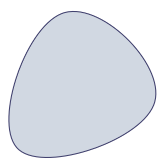
$$f(y) \geq f(x) + \nabla f(x)^\top (y - x) \quad \text{for all } x, y.$$

When $f$ is twice differentiable, $f$ is convex iff $\nabla^2 f(x) \succeq 0$ for all $x$.

**Example (Key convexity facts):** The following will be used throughout the course. These are proven in Lecture 20.

1. *Linear functions are convex*: $f(\theta x + (1-\theta)y) = \theta f(x) + (1-\theta) f(y)$ (equality holds).
2. *Quadratic functions*: $f(x) = \frac{1}{2}x^\top P x + q^\top x + r$ is convex iff $P \succeq 0$, since $\nabla^2 f = P$.
3. *Norms are convex*: $\|\theta x + (1-\theta)y\| \leq \theta\|x\| + (1-\theta)\|y\|$ by the triangle inequality.
4. *Affine composition*: if $f$ is convex then $g(x) = f(Ax+b)$ is convex for any $A, b$.

**Definition**: *Convex set.* A set $C$ is convex if $x_1, x_2 \in C$ implies $\theta x_1 + (1-\theta)x_2 \in C$ for all $0 \leq \theta \leq 1$.



**Example (Convex sets):**

1. *Probability simplex*: $\{x \in \mathbb{R}^n : x \geq 0, \mathbf{1}^\top x = 1\}$ is convex.
2. *PSD matrices*: the set of symmetric positive semidefinite matrices is convex, since $x^\top(\theta_1 X_1 + \theta_2 X_2)x = \theta_1 \underbrace{x^\top X_1 x}_{\geq 0} + \theta_2 \underbrace{x^\top X_2 x}_{\geq 0} \geq 0$.
3. *Sublevel sets*: any $\alpha$-sublevel set $C_\alpha = \{x : f(x) \leq \alpha\}$ of a convex function is convex. (The converse does not hold.)

## *Convex Optimization*

The problem (3) is **convex** if $f_0$ and all $f_i$'s are convex, in which case the feasible set is also convex. Equality constraints are permitted only if affine (of the form $Ax + b = 0$), since requiring both $f_i$ and $-f_i$ to be convex forces $f_i$ to be affine.

**Example (Least squares is a convex QP):** minimize$_x \|Ax - b\|_2^2$: the norm is convex, squaring preserves convexity, and composition with $x \mapsto Ax - b$ (affine) preserves convexity. Expanding gives

$\|Ax - b\|_2^2 = x^\top A^\top Ax - 2b^\top Ax + b^\top b$ with $A^\top A \succeq 0$, confirming it is a QP. Closed-form solution: $x^* = (A^\top A)^{-1} A^\top b$.

**Theorem**: (Optimality condition). *For a convex optimization problem, a feasible point $x$ is optimal if and only if*

$$\nabla f_0(x)^\top (y - x) \geq 0 \quad \text{for all feasible } y.$$

*Proof.* *(if)* By convexity, $f_0(y) \geq f_0(x) + \nabla f_0(x)^\top (y - x) \geq f_0(x)$ for all feasible $y$.

*(only if)* Suppose $x$ is optimal but $\nabla f_0(x)^\top (y - x) < 0$ for some feasible $y$. Then $z_\theta = \theta y + (1 - \theta)x$ is feasible (convex feasible set), and $\frac{d}{d\theta} f_0(z_\theta)|_{\theta=0} = \nabla f_0(x)^\top (y - x) < 0$, so $f_0(z_\theta) < f_0(x)$ for small $\theta$, contradicting optimality. □

For unconstrained problems the condition reduces to $\nabla f_0(x) = 0$. For minimize$_x \frac{1}{2} x^\top Px + q^\top x + r$ with $P \succeq 0$, this gives $Px + q = 0$, with three cases: (1) no solution if $q \notin \text{Range}(P)$; (2) unique solution $x^* = -P^{-1}q$ if $P \succ 0$; (3) affine solution set $\{x^* + y : y \in \text{Null}(P)\}$ if $P$ is singular but $q \in \text{Range}(P)$.

## *Important Classes of Convex Optimization Problems*

### *Linear Programs (LP)*

$$\text{minimize}_x \; c^\top x \quad \text{s.t.} \quad a_i^\top x \leq b_i, \; i = 1, \ldots, m.$$

LPs are solved very efficiently; when the feasible set is compact, an optimal point is attained at a vertex of the feasible region.

### *Quadratic Programs (QP)*

$$\text{minimize}_x \; \tfrac{1}{2} x^\top Px + q^\top x + r \quad \text{s.t.} \quad a_i^\top x \leq b_i, \; i = 1, \ldots, m, \quad P \succeq 0.$$

The CLF controller (1) is a QP (quadratic cost, affine constraint). All LPs are QPs (set $P = 0$).

### *Quadratically Constrained Quadratic Programs (QCQP)*

$$\text{minimize}_x \; \tfrac{1}{2} x^\top P_0 x + q_0^\top x + r_0 \quad \text{s.t.} \quad \tfrac{1}{2} x^\top P_i x + q_i^\top x + r_i \leq 0, \; i = 1, \ldots, m, \quad P_i \succeq 0.$$

All QPs are QCQPs.

*Second-Order Cone Programs (SOCP)*

$$\text{minimize}_x \; f^\top x \quad \text{s.t.} \quad \|A_i x + b_i\|_2 \leq c_i^\top x + d_i, \; i = 1, \ldots, m.$$

Setting all $c_i = 0$ recovers QCQP; setting all $A_i = 0$ recovers LP.

The hierarchy is: LP ⊂ QP ⊂ QCQP ⊂ SOCP ⊂ SDP.

## *Linear Matrix Inequalities and Semidefinite Programs*

Instead of scalar inequalities ($\leq$) in constraints, we can allow matrix inequalities ($\preceq$).

*SDP: First Form*

$$\text{minimize}_x \; c^\top x \quad \text{s.t.} \quad x_1 F_1 + \cdots + x_n F_n + G \preceq 0,$$

where $F_1, \ldots, F_n, G$ are symmetric matrices. This constraint is called a **linear matrix inequality (LMI)**, and the problem is a **semidefinite program (SDP)**. When all $F_i, G$ are scalars, the LMI reduces to an affine inequality (LP).

*The LMI constraint leads to a convex feasible set.* For any two feasible $x$ and $\hat{x}$:

$$\begin{aligned}&(\theta x_1 + (1-\theta)\hat{x}_1)F_1 + \cdots + (\theta x_n + (1-\theta)\hat{x}_n)F_n + G \\ &\quad = \theta(x_1 F_1 + \cdots + x_n F_n + G) + (1-\theta)(\hat{x}_1 F_1 + \cdots + \hat{x}_n F_n + G) \preceq 0.\end{aligned}$$

Multiple LMIs can be combined into one via block diagonalization:

$$x_1 \begin{bmatrix} F_1 & 0 \\ 0 & \hat{F}_1 \end{bmatrix} + \cdots + x_n \begin{bmatrix} F_n & 0 \\ 0 & \hat{F}_n \end{bmatrix} + \begin{bmatrix} G & 0 \\ 0 & \hat{G} \end{bmatrix} \preceq 0.$$

*SDP: Second Form*

$$\text{minimize}_X \; \text{trace}(CX) \quad \text{s.t.} \quad \text{trace}(A_i X) = b_i, \; i = 1, \ldots, m, \quad X \succeq 0.$$

The two forms are equivalent. Matrix variables appearing linearly in semidefinite constraints are permitted.

*LMI Examples*

**Example (Lyapunov inequality):** The map $\mathcal{L}(X) = A^\top X + XA$ is linear in $X$ (verify: $\mathcal{L}(aX_1 + bX_2) = a\mathcal{L}(X_1) + b\mathcal{L}(X_2)$). We know $A$

is Hurwitz iff there exists $X \succ 0$ with $\mathcal{L}(X) \prec 0$. Thus $\mathcal{L}(X) \preceq -\varepsilon I$ is an LMI constraint in the variable $X$.

**Example (Common Lyapunov function for switched systems):**
Consider $\dot{x} = A(t)x$ where $A(t) \in \{A_1, \ldots, A_m\}$. Even if all $A_i$ are Hurwitz, stability of the switched system is not guaranteed. We can search for a common Lyapunov function $V(x) = x^\top P x$ via the SDP:

$$\text{minimize}_P \ \text{trace}(P) \quad \text{s.t.} \quad PA_i + A_i^\top P \preceq -\varepsilon I \ \ \forall i, \quad P \succeq I.$$

## *Solving Convex Optimization Problems in Practice*

Analytic solutions rarely exist, but modern solvers are fast enough that exact solutions are treated as readily available.

Note: As a student, you all have free axis to GITHUB Copilot which can help you try out new coding languages faster by providing help with syntax

- **General-purpose:** CVX (MATLAB), CVXPY (Python), CVXOPT, YALMIP.
- **Specialized:** e.g., MATLAB's `quadprog` for QPs.

**Example (CVXPY: Constrained Least Squares):**

```
import cvxpy as cp

x = cp.Variable(n)
prob = cp.Problem(cp.Minimize(cp.norm(A @ x - b, 2)),
                  [C @ x <= d])
prob.solve()
```

**Example (CVXPY: CLF-QP controller (1)):**

```
import cvxpy as cp

u    = cp.Variable(m)
Lf_V = ...   # partial V/partial x @ f(x)
Lg_V = ...   # partial V/partial x @ g(x)
A_x  = ...   # A(x)

prob = cp.Problem(cp.Minimize(cp.sum_squares(u)),
                  [Lf_V + Lg_V @ u <= -eps * A_x])
prob.solve()
u_opt = u.value
```