ECE 6552 – Lecture 20

A PRIMER ON CONVEX OPTIMIZATION

April 2 2026

Overview:

- Define optimization problems
- Define convex functions and sets
- Define convex optimization problems

Additional Reading:

- S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.

The purpose of this lecture is to dive deeper into optimization problems and convex functions/sets.

Optimization Problems

We often encounter problems of the form

$$\begin{aligned} \text{minimize}_x \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0, \qquad i = 1, \dots, m \end{aligned} \tag{1}$$

Equality constraint $f(x) = 0$ is allowed by including two constraints $f(x) \leq 0$ and $-f(x) \leq 0$

where:

- $x \in \mathbb{R}^n$ is an optimization variable,
- f_0 is the objective function, and
- $f_i(x)$ are constraint functions.

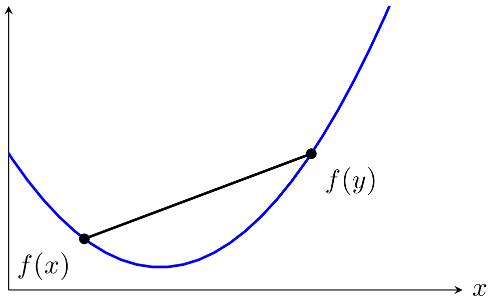
If we are instead interested in maximizing a function $\tilde{f}_0(x)$, we simply define $f_0(x) = -\tilde{f}_0(x)$ to change to a minimization problem.

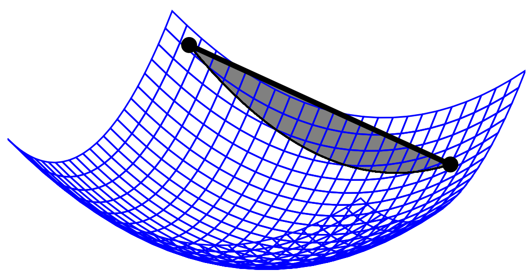
The *optimal value* of $f_0(x)$ is the (limit of the) smallest value obtained by $f_0(x)$ on the *feasible set*. A point that achieves the optimal value (*i.e.*,argmin) is an *optimal point*.

Convex Functions and Sets

Definition: *Convex function.* A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if for all $x, y \in \mathbb{R}^n$ and all $0 \leq \theta \leq 1$:

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y).$$





Example (Linear functions are convex): Consider $f(x) = c^\top x$ for fixed $c \in \mathbb{R}^n$:

$$f(\theta x + (1-\theta)y) = c^\top(\theta x + (1-\theta)y) = \theta c^\top x + (1-\theta)c^\top y = \theta f(x) + (1-\theta)f(y).$$

So f is convex (with equality for all θ).

First- and Second-Order Tests for Convexity

- **First-order test.** When f is once differentiable, f is convex if and only if
$$f(y) \geq f(x) + \nabla f(x)^\top (y-x) \quad \text{for all } x, y.$$
- **Second-order test.** When f is twice differentiable, f is convex if and only if
$$\nabla^2 f(x) \succeq 0 \quad \text{for all } x.$$

Example (Quadratic functions): Consider $f(x) = \frac{1}{2}x^\top Px + q^\top x + r$ with $P = P^\top$. Then $\nabla^2 f(x) = P$ for all x, so f is convex if and only if $P \succeq 0$.

Example (Norms are convex): Any norm $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is convex by the triangle inequality:

$$\|\theta x + (1-\theta)y\| \leq \|\theta x\| + \|(1-\theta)y\| = \theta\|x\| + (1-\theta)\|y\|.$$

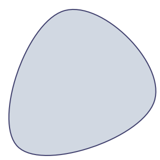
Example (Composition with affine map preserves convexity): If f is convex, then $g(x) = f(Ax+b)$ is convex for any A, b:

$$\begin{aligned} g(\theta x + (1-\theta)y) &= f(A(\theta x + (1-\theta)y) + b) \\ &= f(\theta(Ax+b) + (1-\theta)(Ay+b)) \\ &\leq \theta f(Ax+b) + (1-\theta)f(Ay+b) = \theta g(x) + (1-\theta)g(y). \end{aligned}$$

Convex Sets

Definition: *Convex set.* A set C is convex if whenever $x_1, x_2 \in C$, then

$$\theta x_1 + (1-\theta)x_2 \in C \quad \text{for all } 0 \leq \theta \leq 1.$$



Example (Probability simplex): The probability simplex is the set of vectors $x \in \mathbb{R}^n$ such that $x \geq 0$ and $\mathbf{1}^\top x = 1$. It is convex: for any two elements x_1, x_2 and $0 \leq \theta \leq 1$,

$$\theta x_1 + (1-\theta)x_2 \geq 0 \quad \text{and} \quad \mathbf{1}^\top(\theta x_1 + (1-\theta)x_2) = \theta + (1-\theta) = 1.$$

Example (Positive semidefinite matrices): The set of symmetric positive semidefinite matrices is a convex subset of $\mathbb{R}^{n \times n}$. For any PSD matrices X_1, X_2 and $\theta_1, \theta_2 \geq 0$:

$$x^\top(\theta_1 X_1 + \theta_2 X_2)x = \theta_1 \underbrace{x^\top X_1 x}_{\geq 0} + \theta_2 \underbrace{x^\top X_2 x}_{\geq 0} \geq 0 \quad \text{for all } x.$$

Example (Sublevel sets of convex functions are convex): Any α-sublevel set $C_\alpha = \{x : f(x) \leq \alpha\}$ of a convex function is convex.

Proof. Choose $x, y \in C_\alpha$ so that $f(x) \leq \alpha$ and $f(y) \leq \alpha$. By convexity, $f(\theta x + (1-\theta)y) \leq \alpha$ for any $0 \leq \theta \leq 1$, so $\theta x + (1-\theta)y \in C_\alpha$. □

Note: The converse does not hold.

Convex Optimization

Definition: *Convex optimization problem.* The optimization problem (1) is convex if f_0 and all f_i's are convex. In this case, the feasible set is a convex set.

Example (Equality constraints must be affine): Convex optimization problems may include equality constraints, but only if they are affine, i.e., of the form $Ax + b = 0$.

Proof. To include $f_i(x) = 0$, we add $f_i(x) \leq 0$ and $-f_i(x) \leq 0$. Convexity of both f_i and $-f_i$ requires:

$$f_i(y) \geq f_i(x) + \nabla f_i(x)^\top (y - x) \quad \text{and} \quad -f_i(y) \geq -f_i(x) - \nabla f_i(x)^\top (y - x),$$

so $f_i(y) = f_i(x) + \nabla f_i(x)^\top (y - x)$ for all x, y, i.e., f_i is affine. □

Theorem: Optimality condition. For a convex optimization problem, a feasible point x is optimal if and only if

$$\nabla f_0(x)^\top (y - x) \geq 0 \quad \text{for all feasible } y.$$

Proof. *(if)* Since f_0 is convex, $f_0(y) \geq f_0(x) + \nabla f_0(x)^\top (y - x) \geq f_0(x)$ for all feasible y, so x is optimal.

(only if) Suppose x is optimal but there exists feasible y with $\nabla f_0(x)^\top (y - x) < 0$. The point $z_\theta = \theta y + (1 - \theta)x$ is feasible (the feasible set is convex). For small θ,

$$\frac{d}{d\theta} f_0(z_\theta)\big|_{\theta=0} = \nabla f_0(x)^\top (y - x) < 0,$$

so $f_0(z_\theta) < f_0(x)$, contradicting optimality. □

Optimality for Unconstrained Problems

When all y are feasible, the condition reduces to:

$$x \text{ is optimal} \iff \nabla f_0(x) = 0.$$

Example : Consider $\text{minimize}_x \frac{1}{2} x^\top P x + q^\top x + r$ with $P \succeq 0$. Then x is optimal iff $Px + q = 0$. Three cases:

1. If $q \notin \text{Range}(P)$: no solution; the objective is unbounded below.
2. If $P \succ 0$: unique solution $x^* = -P^{-1} q$.
3. If P is singular but $q \in \text{Range}(P)$: the optimal set is the affine subspace $\{x^* + y \mid y \in \text{Null}(P)\}$, where x^* is any vector with $Px^* = -q$.