ECE 6552 – Lecture 19

CONTROL LYAPUNOV FUNCTIONS

March 31 2026

Overview:

- Normal Form for MIMO Systems
- Define Control Lyapunov Functions
- Present Sontag's Universal Formula for Smooth Stabilization

Additional Reading:

- E. Sontag, 1983
- Z. Arstein, 1978

Normal form for MIMO systems

The notion of zero dynamics and the normal form can be extended to MIMO systems[1]. If the system has vector relative degree $\{r_1, \cdots, r_m\}$, then $r := r_1 + \cdots + r_m \leq n$ and

$$\eta := [h_1(x)\ L_f h_1(x) \cdots L_f^{r_1-1} h_1(x)\ \cdots\ h_m(x)\ L_f h_m(x) \cdots L_f^{r_m-1} h_m(x)]^T$$

defines a partial set of coordinates. As in normal form discussed in Lecture 17, one can find $n-r$ additional functions $z_1(x), \cdots, z_{n-r}(x)$ so that $x \mapsto (z, \eta)$ is a complete coordinate transformation.

Full-state feedback linearization amounts to finding m output functions $h_1, \cdots, h_m$ such that the system has vector relative degree $\{r_1, \cdots, r_m\}$ with $r_1 + \cdots + r_m = n$. Necessary and sufficient conditions for the existence of such functions, analogous to those in Lecture 18 for SISO systems, are available[2].

[1] see, *e.g.*, Sastry, Section 9.3

[2] see, *e.g.*, Sastry, Proposition 9.16

Control Lyapunov Functions

Motivation: Feedback linearization stabilizes systems by "cancelling" the nonlinear dynamics and forcing a system to act like a linear one. While this is better than simply "ignoring" nonlinear dynamics (through classic linearization), it still does not take advantage of the natural dynamics of the system. This fundamental limitation is addressed through the use of *control Lyapunov functions*.

Intro: We had previously utilized Lyapunov for *analysis* of the system

$$\dot{x} = f(x), \qquad f(0) = 0 \tag{1}$$

Here, the goal was to find a positive definite Lyapunov function $V(x)$ such that $\dot{V}(x)$ is negative definite to prove asym. stability of $x = 0$.

What about controlling for (asymptotic) stability?

- An idea: For

$$\dot{x} = f(x) + g(x)u \tag{2}$$

  and a candidate positive definite Lyapunov function $V(x)$, choose u such that $\dot{V}$ is negative definite

Definition: *Control Lyapunov Function.* A positive definition function $V(x)$ is a (global) control Lyapunov function (CLF) for (2) if $\forall x \neq 0$, $\exists\, u$ such that

$$\dot{V}(x) = \frac{\partial V}{\partial x}\left[f(x) + g(x)u\right] < 0. \tag{3}$$

Equivalently,

$$\frac{\partial V}{\partial x} g(x) = 0 \quad \text{and} \quad x \neq 0 \quad \Longrightarrow \quad \frac{\partial V}{\partial x} f(x) < 0. \tag{4}$$

In today's lecture we will introduce a closed-form expression for a CLF, known as Sontag's formula. In the next lecture, we will instead solve for u through convex optimization.

If $u \in \mathbb{R}$, Sontag's formula is:

$$u = \phi(x) = \begin{cases} -\left[\left(\frac{\partial V}{\partial x} f\right) + \sqrt{\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4}\right] / \left(\frac{\partial V}{\partial x} g\right) & \text{if} \quad \frac{\partial V}{\partial x} g \neq 0 \\ 0 & \text{if} \quad \frac{\partial V}{\partial x} g = 0 \end{cases} \tag{5}$$

Note:

- Choosing $u = \phi(x)$ asymptotically stabilizes the origin (Proof is shown next).
- Formula seems complicated. Why? (Examples shown later)

Proof. Compute $\dot{V}(x)$ for $x \neq 0$:

- If $\frac{\partial V}{\partial x} g(x) = 0$, then

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) < 0$$

  for any $x \neq 0$ by definition of CLF

- If $\frac{\partial V}{\partial x} g(x) \neq 0$, then

$$\dot{V}(x) = \frac{\partial V}{\partial x}[f(x) + g(x)\phi(x)] = -\sqrt{\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4} < 0$$

  Therefore, $x \neq 0$ implies $\dot{V}(x) < 0$, which shows asymptotic stability.

□

Example 1:

Consider

$$\dot{x} = -x^3 + u$$

with CLF $V(x) = \frac{1}{2}x^2$. Let's consider the following controllers:

1. $u \equiv 0$:
2. Feedback linearizing controller
3. $u = \phi(x)$ from Sontag's formula

Controller 1:

$$\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x} = x(-x^3) = -x^4 < 0 \quad \text{for} \quad x \neq 0 \tag{6}$$

so the system is globally asymptotically stable but not exponentially stable[3].

Controller 2: With the goal of driving $x \to 0$, we can choose our output $y = x$. This implies that $r = 1$ (since we need to differentiate x once to get to u). The feedback linearizing controller would be

$$u = x^3 + v$$

Therefore, choosing $v = -k_1 x$ for some $k_1 > 0$ yields the closed loop system:

$$\dot{x} = -k_1 x \quad \Longrightarrow \quad \dot{V} = x(-k_1 x) = -k_1 x^2$$

hence the system is now exponentially stable.

Controller 3: To apply Sontag's formula, we need to compute the terms:

$$\frac{\partial V}{\partial x} f(x) = x(-x^3) = -x^4, \quad \frac{\partial V}{\partial x} g(x) = x(1) = x$$

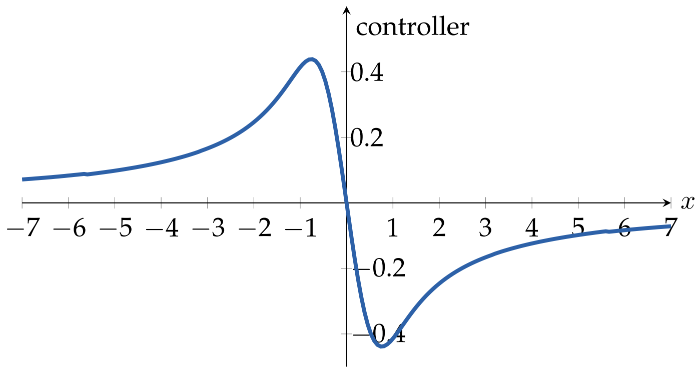
[3] Recall that exponential stability requires a linear bound on $\dot{V}$ in terms of V itself, i.e.,:

$$\dot{V}(x) \leq -cV(x)$$

for some $c > 0$

Plugging these into Sontag's formula yields:

$$\begin{aligned} u &= -1/x \left(-x^4 + \sqrt{x^8 + x^4}\right) \\ &= -1/x \left(-x^4 + x^2\sqrt{x^4+1}\right) \\ &= x^3 - x\sqrt{x^4+1} \end{aligned}$$

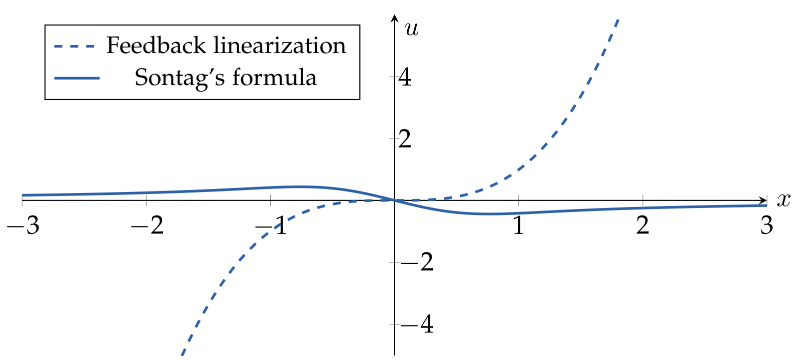


This control law yields the closed-loop system:

$$\begin{aligned} \dot{V}(x) &= x\left(-x^3 + x^3 - x\sqrt{x^4+1}\right) = -x^2\sqrt{x^4+1} \\ &\le -x^2 \end{aligned}$$

where the last inequality follows from the fact that $\sqrt{x^4+1} \ge 1$. Therefore, the system is also globally exponentially stable.

A comparison of the two controllers is shown below:



In general, Sontag's formula can keep useful nonlinearities (like $-x^3$), while feedback linearization cancels all nonlinearities. However, there is no universal theorem that Sontag's formula is always "better".

Example 2:

Consider the system

$$\dot{x} = x - x^3 + u$$

The feedback linearizing controller for this system is:

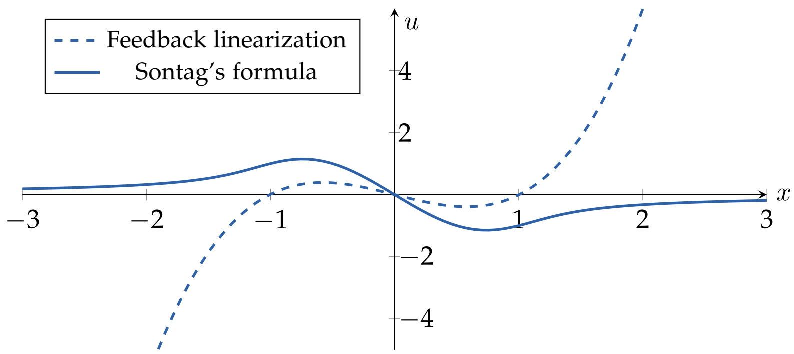
$$u = -x + x^3 - k_1 x$$

for any $k_1 > 0$.

A CLF (using Sontag's formula) for this system is

$$\begin{aligned} u &= \frac{x(-x+x^3) - x\sqrt{x^2(x-x^3)^2 + x^4}}{x} \\ &= -x + x^3 - \sqrt{(1-x^2)^2 + 1} \end{aligned}$$

A comparison of the two controllers is shown below:



Optimization-Based Control Lyapunov Functions

Another approach to stabilizing a system using CLFs is to solve for u through convex optimization. For example, we can solve the following optimization problem at each time step:

$$\begin{aligned} u^* = \phi(x) := \ & \text{argmin}_u \ \|u\|^2 \\ & \text{subject to} \quad \frac{\partial V}{\partial x}(f(x) + g(x)u) \leq -\varepsilon V(x) \end{aligned}$$

This formulation is sometimes called a *control Lyapunov function quadratic program* (CLF-QP). Explicitly, the goal is to find the smallest control input u that stabilizes our system and additionally ensures that the Lyapunov function $V(x)$ decreases at a rate of at least $\varepsilon V(x)$.