ECE 6552 – Lecture 18

FEEDBACK LINEARIZATION 4 (FEEDBACK LINEARIZATION FOR MIMO SYSTEMS)

March 17 2026

Overview:

- Frobenius Theorem continued
- Feedback linearization for MIMO Systems

Additional Reading:

- Khalil, Chapter 13.3

Recap on Full State Feedback Linearization

Recall that our condition for $r = n$ is:

$$L_g h(x) = L_g L_f h(x) = \cdots = L_g L_f^{n-2} h(x) = 0 \text{ in a nbhd of } x_0 \tag{1}$$

$L_{\mathrm{ad}_f g} h(x) = L_{[f,g]} h(x) = \frac{\partial h}{\partial x}\left(\frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x)\right)$

which is equivalent to:

$$L_g h(x) = L_{[f,g]} h(x) = \cdots = L_{[f,[f,\ldots,[f,g]]]} h(x) = 0 \tag{2}$$

which is equivalent to:

$$L_g h(x) = L_{\mathrm{ad}_f g} h(x) = \cdots = L_{\mathrm{ad}_f^{n-2} g} h(x) = 0 \text{ in a nbhd of } x_0 \tag{3}$$

where the benefit is that the $h(x)$ term can be moved outside:

$$\frac{\partial h}{\partial x}\begin{bmatrix} g & \mathrm{ad}_f g & \mathrm{ad}_f^2 g & \ldots & \mathrm{ad}_f^{n-2} g \end{bmatrix} = 0$$

This was the basis for the following necessary and sufficient conditions for feedback linearizability.

Theorem: Full-state Feedback Linearizable. *The system $\dot{x} = f(x) + g(x)u$ is feedback linearizable around x_0 if and only if the following two conditions hold:*

C1) $[g(x_0) \;\; \mathrm{ad}_f g(x_0) \;\; \ldots \;\; \mathrm{ad}_f^{n-1} g(x_0)]$ *has rank* n

C2) $\Delta(x) = \mathrm{span}\{g(x), \mathrm{ad}_f g(x), \ldots, \mathrm{ad}_f^{n-2} g(x)\}$ *is involutive in a neighborhood of* x_0.

Recall that the Frobenius Theorem states that a nonsingular distribution is completely integrable if and only if it is involutive. A nonsingular distribution is completely integrable if there exist $n - r$ functions, h_i, such that

$$\frac{\partial h_i}{\partial x} g_j = 0, \quad j = 1, \ldots, k$$

A distribution is nonsingular if the elements of the span are linearly independent for all x.

Example:

Consider the following system:

$$\begin{aligned}\dot{x}_1 &= x_2 + 2x_1^2\\ \dot{x}_2 &= x_3 + u\\ \dot{x}_3 &= x_1 - x_3\end{aligned}$$

Can we verify our previous choice of $y = h(x) = x_3$ with the theorem above? We will begin by computing the elements of the span[1]:

$$f(x) = \begin{bmatrix} x_2 + 2x_1^2 \\ x_3 \\ x_1 - x_3 \end{bmatrix} \quad g(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$[f,g](x) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \quad [f,[f,g]](x) = \begin{bmatrix} 4x_1 \\ 0 \\ 1 \end{bmatrix}$$

Conditions of the theorem:

1. $\begin{bmatrix} 0 & -1 & 4x_1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ full rank
2. $\Delta = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\right\}$ involutive

   $\dfrac{\partial h}{\partial x}\begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$ satisfied by $h(x) = x_3$.

Example

Let's consider another example:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= u\\ \dot{x}_3 &= x_1 + x_2^2\end{aligned}$$

Again, we begin with the first condition, which requires calculating $\begin{bmatrix} g(x_0) & \text{ad}_f\, g(x_0) \end{bmatrix}$:

Note that

$$f(x) = \begin{bmatrix} x_2 \\ 0 \\ x_1 + x_2^2 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

---

[1] Recall that $[f,g] = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x)$

$$[f,g] = 0 - \begin{bmatrix} 4x_1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$[f,[f,g]] = 0 - \begin{bmatrix} 4x_1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 0 \\ 1 \end{bmatrix}$$

$$
\begin{aligned}
g(x) &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
[f,g](x) &= 0 - \frac{\partial f}{\partial x} g(x) = - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 2x_2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -2x_2 \end{bmatrix} \\
[f,[f,g]](x) &= \frac{\partial [f,g]}{\partial x} f - \frac{\partial f}{\partial x} [f,g] \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ 0 \\ x_1 + x_2^2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 2x_2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ -2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}
$$

So the matrix we need to check for full rank is

$$
\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & \cancelto{0}{2x_2} & 1 \end{bmatrix}
$$

This is full rank. Moving onto the second condition, we check if there exists an output that satisfies:

$$
\frac{\partial h}{\partial x} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 2x_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}
$$

There is no such $h(x)$, so the system is *not* feedback linearizable.

Example

Let's consider another example:

$$
\begin{aligned}
\dot{x}_1 &= -x_1 + x_2^2 + u \\
\dot{x}_2 &= -x_2 + u
\end{aligned}
$$

Checking the first condition:

$$
g(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad [f,g] \quad = 0 - \frac{\partial f}{\partial x} g = - \begin{bmatrix} -1 & 2x_2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - 2x_2 \\ 1 \end{bmatrix}
$$

This gives us the matrix:

$$
[g(0) [f,g](0)] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
$$

This has rank 1, so the first condition fails and the system is *not* feedback linearizable.

Feedback Linearization Continued

A system is a “strict feedback system" if it has the form:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2\\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)x_3\\ \dot{x}_3 &= f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)x_4\\ &\;\;\vdots\\ \dot{x}_n &= f_n(x) + g_n(x)u.\end{aligned} \tag{4}$$

Such systems are feedback linearizable when $g_i(x_1, \ldots, x_i) \neq 0$ near the origin, $i = 1, 2, \cdots, n$, because the relative degree is n with the choice of output $y = h(x) = x_1$:

$$y^{(n)} = L_f^n h(x) + \underbrace{g_1(x_1)g_2(x_1, x_2)\cdots g_n(x)}_{L_g L_f^{n-1}h(x) \neq 0} u.$$

Feedback linearizability is lost when $g_i(0) = 0$ for some i; however, backstepping may be applicable.

Multi-Input Multi-Output Systems

Consider now a MIMO system with m inputs and m outputs:

$$\begin{aligned}\dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x)u_i = f(x) + \begin{bmatrix} g_1(x) & \ldots & g_m(x)\end{bmatrix}\begin{bmatrix} u_1\\ \vdots\\ u_m\end{bmatrix} \\ y_i &= h_i(x), \quad i = 1, \cdots, m.\end{aligned} \tag{5}$$

Let r_i denote the number of times we need to differentiate y_i to hit at least one input. Then,

$$\begin{bmatrix} y_1^{(r_1)}\\ \vdots\\ y_m^{(r_m)}\end{bmatrix} = \underbrace{\begin{bmatrix} L_f^{r_1}h_1(x)\\ \vdots\\ L_f^{r_m}h_m(x)\end{bmatrix}}_{=:\, B(x)} + \underbrace{\begin{bmatrix} L_{g_1}L_f^{r_1-1}h_1(x) & \cdots & L_{g_m}L_f^{r_1-1}h_1(x)\\ \vdots & & \vdots\\ L_{g_1}L_f^{r_m-1}h_m(x) & \cdots & L_{g_m}L_f^{r_m-1}h_m(x)\end{bmatrix}}_{=:\, A(x)}\begin{bmatrix} u_1\\ \vdots\\ u_m\end{bmatrix}.$$

If $A(x)$ is nonsingular, then the feedback law

$$u = A(x)^{-1}(-B(x) + v)$$

input/output linearizes the system, creating m decoupled chains of integrators:

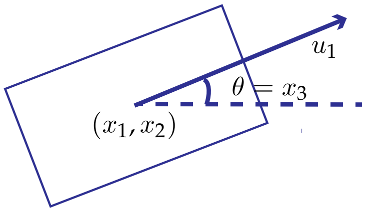
$$y_i^{(r_i)} = v_i, \quad i = 1, \ldots, m.$$

We say that the system has *vector relative degree* $\{r_1, \cdots, r_m\}$ if the matrix $A(x)$ defined above is nonsingular.

Example 2: The kinematic model of a unicycle, depicted below, is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2,$$

where u_1 is the speed and u_2 is the angular velocity.



Let $y_1 = x_1$ and $y_2 = x_2$, and note that

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos x_3 & 0 \\ \sin x_3 & 0 \end{bmatrix}}_{=:\, A(x)} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Since $A(x)$ is singular, the system does not have a well-defined vector relative degree. □

Note that to address the deficit in relative degree, practitioners often add an integrator to the system. This is known as a "dynamic extension" which introduces a new state variable $x_4 = u_1$ with the new inputs becoming $\tilde{u}_1 = \dot{u}_1$ and $u_2 = \dot{\theta}$, yielding:

$$\dot{x} = \begin{bmatrix} x_4 \cos(x_3) \\ x_4 \sin(x_3) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ u_2 \end{bmatrix}$$

Given the same outputs ($y_1 = x_1$, and $y_2 = x_2$), we now have relative degree:

$$\begin{aligned} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} &= \begin{bmatrix} x_4 \cos(x_3) \\ x_4 \sin(x_3) \end{bmatrix} \\ \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} &= \begin{bmatrix} \dot{x}_4 \cos(x_3) - x_4 \dot{x}_3 \sin(x_3) \\ \dot{x}_4 \sin(x_3) + x_4 \dot{x}_3 \cos(x_3) \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} \cos(x_3) & -x_4 \sin(x_3) \\ \sin(x_3) & x_4 \cos(x_3) \end{bmatrix}}_{A(x)} \begin{bmatrix} \tilde{u}_1 \\ u_2 \end{bmatrix} \end{aligned}$$

Thus the system now has a well-defined relative degree as long as $x_4 \neq 0$ (unicycle must always be moving forward to be controllable). This system would have vector relative degree $\{2, 2\}$.

Normal form for MIMO systems

The notion of zero dynamics and the normal form can be extended to MIMO systems[2]. If the system has vector relative degree $\{r_1, \cdots, r_m\}$, then $r := r_1 + \cdots + r_m \leq n$ and

$$\eta := [h_1(x)\ L_f h_1(x) \cdots L_f^{r_1-1} h_1(x)\ \cdots\ h_m(x)\ L_f h_m(x) \cdots L_f^{r_m-1} h_m(x)]^T$$

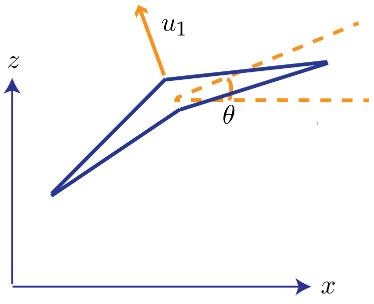
defines a partial set of coordinates. As in normal form discussed in Lecture 17, one can find $n - r$ additional functions $z_1(x), \cdots, z_{n-r}(x)$ so that $x \mapsto (z, \eta)$ is a complete coordinate transformation.

Full-state feedback linearization amounts to finding m output functions $h_1, \cdots, h_m$ such that the system has vector relative degree $\{r_1, \cdots, r_m\}$ with $r_1 + \cdots + r_m = n$. Necessary and sufficient conditions for the existence of such functions, analogous to those in Lecture 18 for SISO systems, are available[3].

Example 3: Consider the following model of a *planar vertical take-off and landing* (PVTOL) aircraft[4]

$$\begin{aligned}\ddot{x} &= -\sin(\theta)u_1 + \mu\cos(\theta)u_2\\ \ddot{z} &= \cos(\theta)u_1 + \mu\sin(\theta)u_2 - 1\\ \ddot{\theta} &= u_2,\end{aligned}$$

where μ is a constant that accounts for the coupling between the rolling moment and translational acceleration, and -1 in the second equation is the gravitational acceleration, normalized to unity by appropriately scaling the variables.



---

[2] see, *e.g.*, Sastry, Section 9.3

[3] see, *e.g.*, Sastry, Proposition 9.16

[4] Sastry, Section 10.4.2

If we take x and z as the two outputs we get

$$\begin{bmatrix}\ddot{x}\\ \ddot{z}\end{bmatrix} = \begin{bmatrix}0\\ -1\end{bmatrix} + \underbrace{\begin{bmatrix}-\sin\theta & \mu\cos\theta\\ \cos\theta & \mu\sin\theta\end{bmatrix}}_{A(\theta)}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

where $A(\theta)$ is invertible when $\mu \neq 0$:

$$A^{-1}(\theta) = \begin{bmatrix}-\sin\theta & \cos\theta\\ \frac{1}{\mu}\cos\theta & \frac{1}{\mu}\sin\theta\end{bmatrix}.$$

Thus the systems has vector relative degree $\{2, 2\}$ This implies that when $\mu \neq 0$, and the input/output linearizing controller is

$$\begin{bmatrix}u_1\\ u_2\end{bmatrix} = \begin{bmatrix}-\sin\theta & \cos\theta\\ \frac{1}{\mu}\cos\theta & \frac{1}{\mu}\sin\theta\end{bmatrix}\left(\begin{bmatrix}0\\ 1\end{bmatrix} + \begin{bmatrix}v_1\\ v_2\end{bmatrix}\right).$$

The zero dynamics is obtained by substituting $u_2^* = \frac{1}{\mu}\sin\theta$, needed to maintain z at a constant value and $\dot{z}$ at zero, in the dynamical equation for θ:

$$\ddot{\theta} = \frac{1}{\mu}\sin\theta.$$

The system is nonminimum phase for $\mu > 0$, since $\theta = 0$ is unstable.